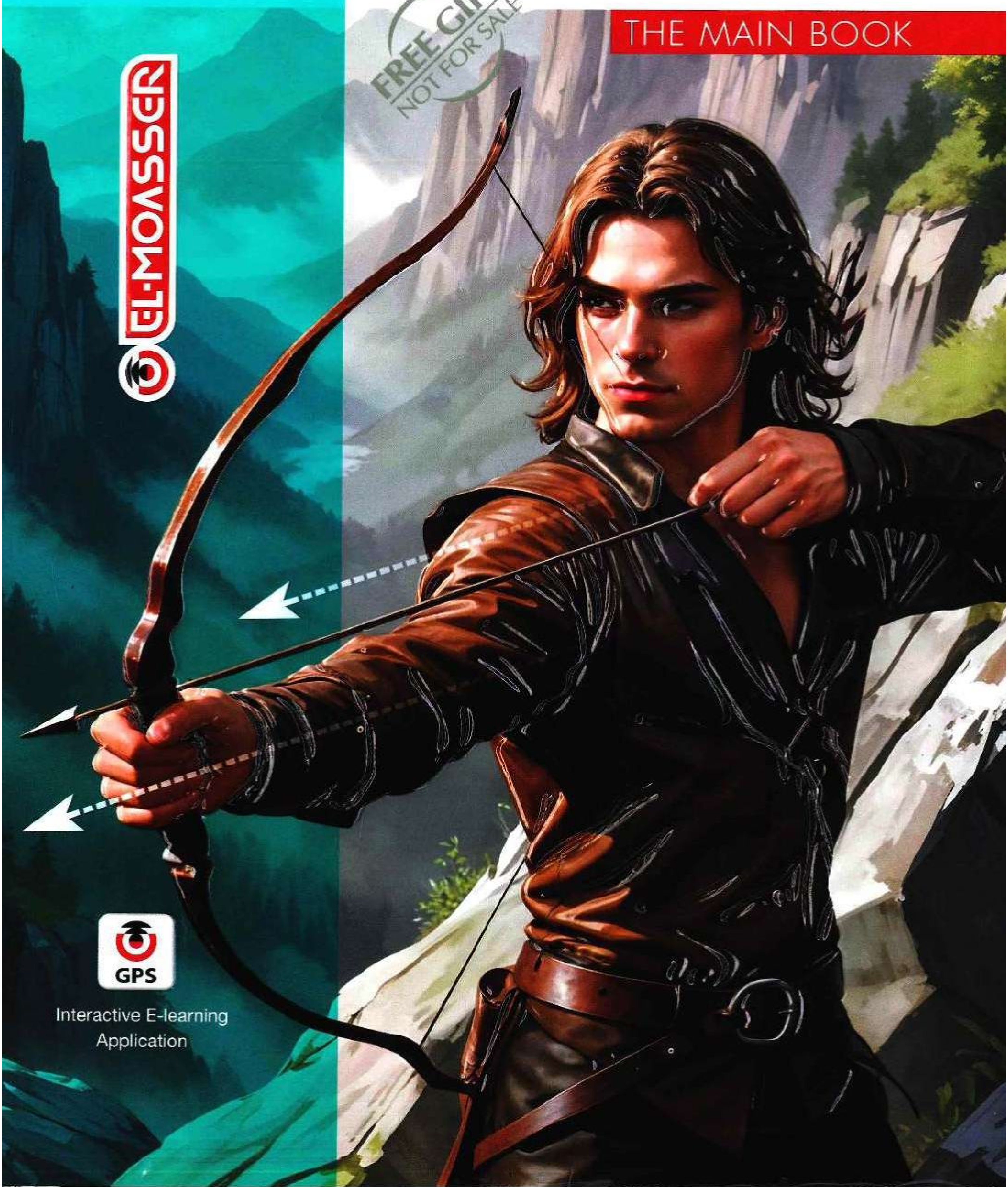
1st Secondary
SECOND TERM

Mathematics

2025

By a group of supervisors

THE MAIN BOOK



CONTENTS

First

Algebra and Trigonometry

Unit **1** Matrices.

Unit **2** Linear programming.

Unit **3** Trigonometry.

Unit 1

Matrices

Unit Lessons

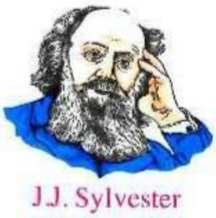
Learning outcomes

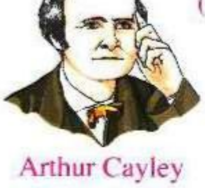
By the end of this unit, the student should be able to :

- Recognize the concept of the matrix and its order.
- Model some real life problems using the matrices.
- Recognize some special matrices.
- Recognize the equality of two matrices.
- Find the transpose of a matrix.
- Multiply a real number by a matrix.
- Recognize the concept of symmetric matrix and skew symmetric matrix.
- Carry out the operations of addition , subtraction and multiplication on matrices.
- Recognize the properties of addition and multiplication of the matrices.
- Use matrices in other domains.
- Recognize the determinant of a matrix of order 2×2 and 3×3
- Find the value of the determinant of order 2×2 and 3×3
- Find the surface area of the triangle using the determinants.
- Solve a system of linear equations using Cramar's rule.
- Find the inverse of the square matrix of order 2×2
- Solve two simultaneous equations using the inverse of a matrix.

Brief History

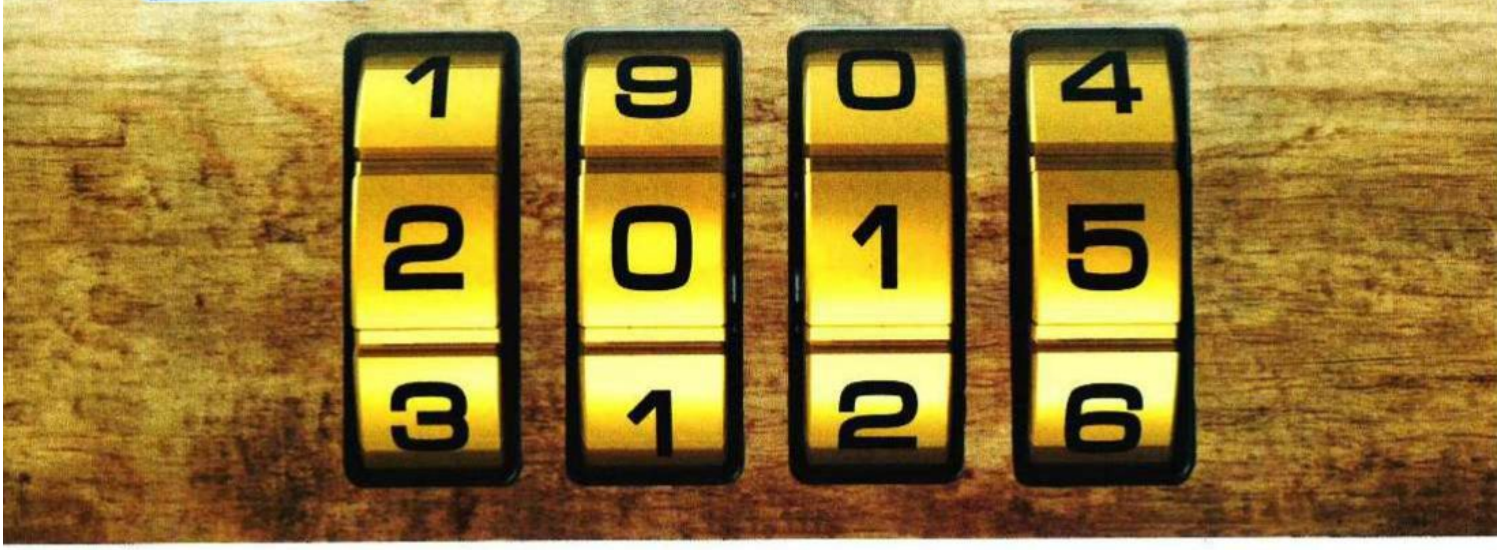
- The British scientist J.J. Sylvester was the first to use the expression "matrix".
- The British scientist Cayley was the first to use the matrices, and he is a mathematical scientist and has a lot of searches especially in algebra which included the theorem of matrix.
- Matrix is commonly used in modern times, and it includes many branches of science and knowledge, we use it in statistics and economics, sociology, psychology and so on. Further more, matrices have an important role in mathematics especially in the branch of linear algebra.


J.J. Sylvester
(1814 - 1897)


Arthur Cayley
(1821 - 1895)

Lesson **1**

Organizing data in matrices



Illustrated example

- A pizzeria sells four kinds of pizzas : (Vegetarian - Chicken - Beef - Cheese) and serves for each kind three different sizes : (small - medium - large)

- To remember these data and compare between them easily , the shop owner arranged the average of the number of sold pieces daily in the following table :

		Size		
		Small	**Medium**	**Large**
Kind	**Vegetarian**	15	13	9
	Chicken	16	18	12
	Beef	13	10	8
	Cheese	18	20	17

- Each number in this table has a certain meaning , for example , the number 10 refers to the number of sold pieces of beef with medium size and the number 12 refers to the number of sold pieces of chicken with large size and so on.

- For we know that the numbers of the first row refer to the average of sold pieces of vegetarian daily with the sizes : (small - medium - large) respectively , similarly , the numbers of the second row for chicken , the numbers of the third row for beef and the numbers of the fourth row for cheese respectively , then we don't need the previous table , and we are satisfied to write the data in a simple form by writing only the contained numbers in the table in the same order inside two large parentheses as $\begin{pmatrix} & \end{pmatrix}$

Then we write the daily averages for the sales of the shop = $\begin{pmatrix} 15 & 13 & 9 \\ 16 & 18 & 12 \\ 13 & 10 & 8 \\ 18 & 20 & 17 \end{pmatrix}$

- This form is called a **matrix** and the numbers contained inside the two parentheses are called **the elements of the matrix**.
- This matrix is formed from four rows and three columns as in the opposite figure , so we say that it is a matrix of order 4×3 or simply "a 4×3 matrix" and we notice that we mention the number of rows firstly , then the number of columns.

$$\begin{array}{c c}
\begin{array}{ccc} 1^{st}\text{ column} & 2^{nd}\text{ column} & 3^{rd}\text{ column} \\ \downarrow & \downarrow & \downarrow \end{array} & \\
\begin{pmatrix} 15 & 13 & 9 \\ 16 & 18 & 12 \\ 13 & 10 & 8 \\ 18 & 20 & 17 \end{pmatrix} & \begin{array}{l} \leftarrow 1^{st}\text{ row} \\ \leftarrow 2^{nd}\text{ row} \\ \leftarrow 3^{rd}\text{ row} \\ \leftarrow 4^{th}\text{ row} \end{array}
\end{array}$$

Remark

The owner of the shop can organize his previous data in another table as the following table :

		Kind			
		Vegetarian	**Chicken**	**Beef**	**Cheese**
Size	**Small**	15	16	13	18
	Medium	13	18	10	20
	Large	9	12	8	17

Similarly , we don't need the previous table and we are satisfied with writing the numbers inside a matrix , then the daily averages for the sales of the

shop = $\begin{pmatrix} 15 & 16 & 13 & 18 \\ 13 & 18 & 10 & 20 \\ 9 & 12 & 8 & 17 \end{pmatrix}$ and it is a matrix of order 3×4

From the previous , we can define the matrix as follows :

Definition of the matrix

- The matrix is an arrangement of a number of elements (variables or numbers) in rows and columns enclosed by two parentheses as ( ) , such that the position of each element in the matrix has a meaning.
- If the number of rows is m and the number of columns is n , then the form of the matrix is $m \times n$ or of type $m \times n$ (it is read as m by n) where m and n are positive integers.
- The number of the elements of the matrix = number of rows × number of columns
$= m \times n$

Expressing an element inside a matrix

- Capital letters are used to name the matrix or to symbolize it as : A , B , C , X , Y , ... , but small letters are used to name the elements of the matrix as : a , b , c , x , y , ...
- If we want to express an element inside the matrix A that lies in the i^{th} row and the j^{th} column , then we can express it by the form : a_{ij}

For example :

The element a_{23} lies in the 2^{nd} row and the 3^{rd} column (it is read as : a two three)

The element a_{32} lies in the 3^{rd} row and the 2^{nd} column (it is read as : a three two)

Example 1

If $A = \begin{pmatrix} -2 & 0 & \frac{1}{2} \\ 1 & 6 & -4 \end{pmatrix}$ **,** $B = \begin{pmatrix} 4 & 5 \\ -1 & 8 \end{pmatrix}$ **and** $C = \begin{pmatrix} 3 & -3 & \sqrt{5} \\ -1 & 0 & -2 \\ 2 & \frac{1}{4} & 9 \end{pmatrix}$

1 Write the order of each of A , B and C

2 Write the following elements : a_{22} , b_{21} , a_{23} , b_{12} , c_{32} , c_{23}

Solution

1 A is a matrix of order 2×3 , B is a matrix of order 2×2 and C is a matrix of order 3×3

2 $a_{22} = 6$, $b_{21} = -1$, $a_{23} = -4$, $b_{12} = 5$, $c_{32} = \frac{1}{4}$, $c_{23} = -2$

TRY TO SOLVE

If the matrix $X = \begin{pmatrix} 5 & -2 \\ 4 & 1 \\ -7 & 0 \end{pmatrix}$

1 Write the order of the matrix X.

2 Write the following elements x_{32} , x_{21} , x_{12}

Remark

If A is a matrix of order $m \times n$, then we can express it by the form :

$A = (a_{ij})$ where :

$i = 1 , 2 , 3 , \dots , m$ and $j = 1 , 2 , 3 , \dots , n$

In our study , the cases in which $1 \le m \le 3$ and $1 \le n \le 3$ will be sufficient.

Example 2

Write the matrix (a_{ij}) of order 3×2 where $a_{ij} = 2i - j$

Solution

$\because$ The matrix of order 3×2

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$a_{11} = 2 \times 1 - 1 = 1$, $a_{12} = 2 \times 1 - 2 = 0$, $a_{21} = 2 \times 2 - 1 = 3$,

$a_{22} = 2 \times 2 - 2 = 2$, $a_{31} = 2 \times 3 - 1 = 5$, $a_{32} = 2 \times 3 - 2 = 4$ $\quad \therefore (a_{ij}) = \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{pmatrix}$

Some special matrices

1 The row matrix

It is a matrix that consists of one row and any number of columns.

For example :

$A = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}$ is a row matrix of order 1×3

2 The column matrix

It is a matrix that consists of one column and any number of rows.

For example :

$X = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a column matrix of order 2×1

3 The square matrix

It is a matrix in which the number of rows = the number of columns.

For example :

$A = \begin{pmatrix} \sqrt{3} & 5 \\ -2 & 6 \end{pmatrix}$ is a square matrix of order 2×2

4 The zero matrix

It is a matrix whose all elements are zeroes. We denote it by O_{mn} and can be of any order.

For example :

$O_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero matrix of order 2×2

, $O_{3 \times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is a zero matrix of order 3×1

5 The diagonal matrix

It is a square matrix in which all elements are zeroes except the elements of its main diagonal , then at least one of them is not equal to zero. (Where the main diagonal that contains the elements a_{11} , a_{22} , a_{33} , ...)

For example :

$Y = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a diagonal matrix of order 3×3

, $R = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix of order 2×2

6 The unit matrix

It is a diagonal matrix in which each element on the main diagonal is the number 1 , and it is denoted by I

For example : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a unit matrix of order 2×2

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix of order 3×3

The equality of two matrices

The two matrices A and B are said to be equal if and only if , they have the same order and their corresponding elements are equal.

i.e. $a_{ij} = b_{ij}$ for each i and j

For example : $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ \frac{4}{2} & 3 & -1 \end{pmatrix}$

while $\begin{pmatrix} 2 & 8 \\ -3 & 5 \end{pmatrix} \neq \begin{pmatrix} 2 & 5 \\ -3 & 8 \end{pmatrix}$ because the corresponding elements are different.

and so $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ because they don't have the same order.

Example 3

Find the value of each of X , y and z if $\begin{pmatrix} z & 0 & 2 \\ 4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & X+5 \\ 4 & 2y-3 & 5 \end{pmatrix}$

Solution

$\because$ The two matrices are equal. $\quad \therefore z = -1$

, $X + 5 = 2 \quad \therefore X = -3$, $2y - 3 = 7 \quad \therefore y = 5$

TRY TO SOLVE

Find the value of each of X and y if $\begin{pmatrix} X^3 & -2 \\ 3 & 2y-9 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ 3 & -y \end{pmatrix}$

Multiplying a real number by a matrix

If A is a matrix of order $m \times n$, then the product of any real number k by the matrix A is the matrix $C = k A$ of the same order $m \times n$, and each element of the elements of the matrix C equals the corresponding element to it in the matrix A multiplied by the real number k

i.e. $c_{ij} = k\, a_{ij}$, where $i = 1 , 2 , \dots , m$ and $j = 1 , 2 , \dots , n$

i.e. Multiplying a real number by a matrix means multiplying each element of the elements of the matrix by that real number, and does not change the order of the matrix.

For example :

If $A = \begin{pmatrix} 6 & -2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$, then :

- $2A = \begin{pmatrix} 6 \times 2 & -2 \times 2 & 3 \times 2 \\ 2 \times 2 & 4 \times 2 & 0 \times 2 \end{pmatrix} = \begin{pmatrix} 12 & -4 & 6 \\ 4 & 8 & 0 \end{pmatrix}$
- $-3A = \begin{pmatrix} -18 & 6 & -9 \\ -6 & -12 & 0 \end{pmatrix}$

Remark

From the previous , we deduce that it is possible to take a common factor among all the elements of the matrix.

For example : $\begin{pmatrix} 4 & 8 & 6 \\ 2 & 0 & -14 \end{pmatrix} = 2\begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & -7 \end{pmatrix}$

Example 4

If $\begin{pmatrix} 10 & -20 \\ -8 & 16 \end{pmatrix} = -2\begin{pmatrix} -5 & 5x \\ 4 & -2y \end{pmatrix}$**, then find the value of :** $\sqrt[3]{xy}$

Solution

$\because \begin{pmatrix} 10 & -20 \\ -8 & 16 \end{pmatrix} = \begin{pmatrix} 10 & -10x \\ -8 & 4y \end{pmatrix}$ $\quad \therefore -20 = -10x \quad \therefore x = 2$

$, 16 = 4y \quad \therefore y = 4 \quad \therefore \sqrt[3]{xy} = \sqrt[3]{8} = 2$

TRY TO SOLVE

1 If $A = \begin{pmatrix} 3 & -4 \\ -1 & 0 \end{pmatrix}$, **then find :** $3A , -A , -5A$

2 If $\begin{pmatrix} 24 & -8 \\ 32 & 0 \\ 12 & -4 \end{pmatrix} = 4\begin{pmatrix} 6 & 2y \\ -2x & 0 \\ 3 & -1 \end{pmatrix}$, **then find :** xy

Matrix transpose

In any matrix A of order $m \times n$, if we replace the rows by columns or the columns by rows in the same order , then we will get a matrix of order $n \times m$ that is called the transpose of the matrix A and we denote it by A^t

i.e. If $A = (a_{ij})$, then $A^t = (a_{ji})$

For example :

Notice that

$\left(A^t\right)^t = A$

- If $A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 6 \end{pmatrix}$ is a matrix of order 2×3

, then $A^t = \begin{pmatrix} 2 & -1 \\ 3 & 5 \\ 0 & 6 \end{pmatrix}$ is a matrix of order 3×2 , $\left(A^t\right)^t = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 6 \end{pmatrix}$

- If $B = \begin{pmatrix} 9 \\ -2 \\ 4 \end{pmatrix}$ is a matrix of order 3×1 (column matrix)

, then $B^t = \begin{pmatrix} 9 & -2 & 4 \end{pmatrix}$ is a matrix of order 1×3 (row matrix) , $\left(B^t\right)^t = \begin{pmatrix} 9 \\ -2 \\ 4 \end{pmatrix} = B$

Example 5

If $A = \begin{pmatrix} \cot 30° & \sec 30° \\ \csc 30° & \sin 30° \end{pmatrix}$, $B = \begin{pmatrix} \sqrt{3} & \frac{1}{2}y \\ \sqrt{3}\,x & \frac{1}{2} \end{pmatrix}$, and $A = B^t$

, then find the value of each of : x , y

Solution

$\because A = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 2 & \frac{1}{2} \end{pmatrix}$, $B^t = \begin{pmatrix} \sqrt{3} & \sqrt{3}\,x \\ \frac{1}{2}y & \frac{1}{2} \end{pmatrix}$, $\because A = B^t$

$\therefore \sqrt{3}\,x = \frac{2}{\sqrt{3}}$ $\therefore x = \frac{2}{3}$, $\frac{1}{2}y = 2$ $\therefore y = 4$

TRY TO SOLVE

If $\begin{pmatrix} 2 & x+2 & -5 \\ 0 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 6 \\ -5 & y \end{pmatrix}^t$ **, then find : $\frac{x}{y}$**

Symmetric and skew symmetric matrices

If A is a square matrix , then :

- A is called a symmetric matrix if and only if $A = A^t$
- A is called a skew symmetric matrix if and only if $A = -A^t$

For example :

- If $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix}$, then $A^t = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix}$

i.e. A is a symmetric matrix because $A = A^t$

- If $B = \begin{pmatrix} 0 & -\frac{1}{2} & -4 \\ \frac{1}{2} & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix}$

, then $B^t = \begin{pmatrix} 0 & \frac{1}{2} & 4 \\ -\frac{1}{2} & 0 & -2 \\ -4 & 2 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -\frac{1}{2} & -4 \\ \frac{1}{2} & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix}$

i.e. B is a skew symmetric matrix because $B = -B^t$

Remarks

- If A is a symmetric matrix , we notice that its elements are symmetric about the main diagonal , then $a_{ij} = a_{ji}$ as in the opposite figure , where $a_{21} = a_{12} = d$, $a_{31} = a_{13} = e$, $a_{32} = a_{23} = f$

$$\begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}$$

The main diagonal

- Any diagonal matrix is a symmetric matrix.
- The unit matrix is a symmetric matrix. ($I^t = I$)
- The elements of the main diagonal in the skew symmetric matrix have the numeral zero , and its elements satisfy the relation $a_{ij} = -a_{ji}$ as in the opposite figure.

$$\begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}$$

The main diagonal

where $a_{21} = -a_{12} = -d$, $a_{31} = -a_{13} = -e$, $a_{32} = -a_{23} = -f$

Example 6

1 If $A = \begin{pmatrix} 5 & 2x & 8 \\ -4 & -3 & 6 \\ x + 2y & 6 & 4 \end{pmatrix}$ **is a symmetric matrix , then find the values of : x , y**

2 **If** $B = \begin{pmatrix} 0 & 3x & 7 \\ z+3 & 0 & -2z \\ 3y-x & 6 & 0 \end{pmatrix}$ **is a skew symmetric matrix**

, then find the values of : x **,** y **,** z

Solution

1 $\because$ A is a symmetric matrix

$\therefore 2x = -4$ $\qquad\qquad$ $\therefore x = -2$

$, x + 2y = 8$ $\qquad\qquad$ $\therefore -2 + 2y = 8$

$\therefore y = 5$

2 $\because$ B is a skew symmetric matrix

$\therefore z + 3 = -3x$ $\quad$ (1) $\qquad$ $, 3y - x = -7$ $\quad$ (2)

$, -2z = -6$ $\qquad\qquad$ $\therefore z = 3$

Substituting in (1) : $\therefore 3 + 3 = -3x$ $\qquad$ $\therefore x = -2$

Substituting in (2) : $\therefore 3y + 2 = -7$ $\qquad$ $\therefore y = -3$

TRY TO SOLVE

Complete the following :

1 If $A = \begin{pmatrix} 5 & 8 \\ -2x & 6 \end{pmatrix}$ is a symmetric matrix , find the value of : x

2 If $B = \begin{pmatrix} 0 & -8 & 5 \\ \frac{1}{2}x & 0 & 12 \\ -5 & y-x & 0 \end{pmatrix}$ is a skew symmetric matrix , find the values of : x , y

Lesson 2

Adding and subtracting matrices

First Adding matrices

If A and B are two matrices of the same order , then the addition operation is possible and the result of addition is a matrix of the same order and each of its elements is the sum of the two corresponding elements in A and B

For example :

If $A = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$, then $A + B = \begin{pmatrix} 2+5 & 1+2 \\ 3+2 & -2+3 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 5 & 1 \end{pmatrix}$

Example 1

If $A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -2 \\ 5 & 1 \\ 3 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix}$

Find each of the following if it is possible : 1 $2A + C^t$ **2** $B + C$

Solution

1 $2A + C^t = 2\begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix}^t$

$$= \begin{pmatrix} 2 & -4 \\ 6 & 10 \\ 8 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 0 & 4 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 6 & 14 \\ 7 & 10 \end{pmatrix}$$

2 It is impossible to add B and C , because they don't have the same order , since B is of order 3×2 and C is of order 2×3

Example 2

If $A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 6 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ 2 & -3 \\ 4 & -8 \end{pmatrix}$ **, check that :** $(A + B)^t = A^t + B^t$

Solution

$\because A + B = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 2 & -3 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 1 & 2 \\ 10 & -1 \end{pmatrix}$

$\therefore (A + B)^t = \begin{pmatrix} 2 & 1 & 10 \\ 7 & 2 & -1 \end{pmatrix}$ (1)

$\because A^t = \begin{pmatrix} 2 & -1 & 6 \\ 3 & 5 & 7 \end{pmatrix}$, $B^t = \begin{pmatrix} 0 & 2 & 4 \\ 4 & -3 & -8 \end{pmatrix}$

$\therefore A^t + B^t = \begin{pmatrix} 2 & -1 & 6 \\ 3 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 4 & -3 & -8 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 10 \\ 7 & 2 & -1 \end{pmatrix}$ (2)

From (1) and (2) , we deduce that : $(A + B)^t = A^t + B^t$

TRY TO SOLVE

If $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -3 \\ -1 & 8 \end{pmatrix}$ **, then find :** $\frac{1}{3}\left(A + B^t\right)$

Example 3

Find the values of a , b and c that satisfy the equation :

$$3\begin{pmatrix} a & b \\ c & 3 \end{pmatrix} = 2\begin{pmatrix} a & 6 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & b+4 \\ c+3 & 3 \end{pmatrix}$$

Solution

$\begin{pmatrix} 3a & 3b \\ 3c & 9 \end{pmatrix} = \begin{pmatrix} 2a & 12 \\ -2 & 6 \end{pmatrix} + \begin{pmatrix} 4 & b+4 \\ c+3 & 3 \end{pmatrix}$ (multiplying a real number by a matrix)

$\therefore \begin{pmatrix} 3a & 3b \\ 3c & 9 \end{pmatrix} = \begin{pmatrix} 2a+4 & b+16 \\ c+1 & 9 \end{pmatrix}$

and from the property of equality of two matrices :

$\therefore 3a = 2a + 4$ $\qquad \therefore a = 4$

$, 3b = b + 16$ $\qquad \therefore b = 8$

$, 3c = c + 1$ $\qquad \therefore c = \frac{1}{2}$

Unit 1

TRY TO SOLVE

If $3\begin{pmatrix} a & b \\ c & -2 \end{pmatrix} = 2\begin{pmatrix} a+1 & 3 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4+b \\ c+3 & -10 \end{pmatrix}$

, then find the value of each of : a , b and c

Properties of adding matrices

Let A , B and C be three matrices of the order m × n and **O** is a zero matrix of the same order , then the following properties will be satisfied :

1 Closure property : A + B is a matrix of the same order m × n

2 Commutative property : $A + B = B + A$

For example :

$$\begin{pmatrix} 4 & 2 \\ -1 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -3 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -3 & 2 \end{pmatrix}$$

3 Associative property : $(A + B) + C = A + (B + C)$

For example :

If $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 & 4 \\ 7 & 8 & -9 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix}$

, then $(A + B) + C = \left[\begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 4 \\ 7 & 8 & -9 \end{pmatrix}\right] + \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -1 & 7 \\ 11 & 13 & -15 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 17 & 20 & 1 \end{pmatrix}$$

, $A + (B + C) = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} + \left[\begin{pmatrix} -2 & 1 & 4 \\ 7 & 8 & -9 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -5 \\ 6 & 7 & 16 \end{pmatrix}\right]$

$$= \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} + \begin{pmatrix} -1 & 4 & -1 \\ 13 & 15 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 17 & 20 & 1 \end{pmatrix}$$

i.e. $(A + B) + C = A + (B + C)$

4 The existence of the additive identity :

Zero matrix O is the additive identity "neutral"

i.e. $\boxed{A + \mathbf{O} = \mathbf{O} + A = A}$

For example :

$$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

5 The existence of the additive inverse :

$\boxed{A + (-A) = (-A) + A = \mathbf{O}}$ where $(-A)$ is the additive inverse of the matrix A

For example :

If $A = \begin{pmatrix} 4 & 1 & 0 \\ -3 & 2 & 5 \end{pmatrix}$, then the additive inverse of A is $-A = \begin{pmatrix} -4 & -1 & 0 \\ 3 & -2 & -5 \end{pmatrix}$

where

$$\begin{pmatrix} 4 & 1 & 0 \\ -3 & 2 & 5 \end{pmatrix} + \begin{pmatrix} -4 & -1 & 0 \\ 3 & -2 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{O}_{2 \times 3}$$

Second Subtracting matrices

If A and B are two matrices of the same order $m \times n$, then the remainder of subtracting $(A - B)$ is the matrix C of the order $m \times n$ that is defined as follows :
$C = A - B = A + (-B)$ where $(-B)$ is the additive inverse of the matrix B

For example :

If $A = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 5 & 0 \end{pmatrix}$

, then $A - B = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} -4 & -3 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & -4 \end{pmatrix}$

Remark

We can carry out the subtraction operation directly by subtracting the corresponding elements of the two matrices.

For example :

$$\begin{pmatrix} 3 & 4 & 5 \\ -2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} -3 & -2 & 6 \\ 0 & -1 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -1 \\ -2 & 2 & -8 \end{pmatrix}$$

Example 4

If $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 4 \\ -2 & -6 \\ 8 & -2 \end{pmatrix}$

, find the value of : $4A - 2B + \frac{1}{2}C$

Solution

$$4A - 2B + \frac{1}{2}C = 4\begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 4 \end{pmatrix} - 2\begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 4 \\ -2 & -6 \\ 8 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 4 \\ -4 & 12 \\ 0 & 16 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 2 & -10 \\ -6 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -1 & -3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -1 \\ -2 & 21 \end{pmatrix}$$

TRY TO SOLVE

If $A = \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

, then find the value of : $2A + 3C - 2B$

Remark

Subtracting matrices operation is not commutative and not associative.

Example 5

If $A = \begin{pmatrix} -1 & 1 & 4 \\ 2 & 5 & 0 \\ 3 & 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 0 & -1 & -3 \end{pmatrix}$

, find the matrix X such that : $3X + 2B = A$

Solution

$\because 3X + 2B = A$ (Add the additive inverse of the matrix (2 B) to both sides)

$\therefore 3X + 2B + (-2B) = A + (-2B)$

$\therefore 3X = A - 2B$ $\left(\text{Multiply the two sides by } \frac{1}{3}\right)$

$\therefore X = \frac{1}{3}(A - 2B)$

$$\therefore X = \frac{1}{3}\left[\begin{pmatrix} -1 & 1 & 4 \\ 2 & 5 & 0 \\ 3 & 7 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 0 & -1 & -3 \end{pmatrix}\right]$$

$$\therefore X = \frac{1}{3}\begin{pmatrix} -3 & -3 & -2 \\ 6 & -3 & 0 \\ 3 & 9 & 8 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -\frac{2}{3} \\ 2 & -1 & 0 \\ 1 & 3 & \frac{8}{3} \end{pmatrix}$$

Example 6

If $A = \begin{pmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix}$

, find the matrix X that satisfies : $2\left[X^t - A\right] = 3\,B$

Solution

$\because 2\left[X^t - A\right] = 3\,B$

$\therefore 2\,X^t - 2\,A = 3\,B$ (Add the matrix 2 A to both sides)

$\therefore 2\,X^t - 2\,A + 2\,A = 3\,B + 2\,A$

$$\therefore 2\,X^t = 3\,B + 2\,A = 3\begin{pmatrix} 0 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix} + 2\begin{pmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 5 \\ 13 & 14 & -2 \end{pmatrix}$$

$$\therefore X^t = \frac{1}{2}\begin{pmatrix} 4 & 3 & 5 \\ 13 & 14 & -2 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{2} \\ \frac{13}{2} & 7 & -1 \end{pmatrix}$$ (Take the transpose of both sides)

$$\therefore (X^t)^t = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{2} \\ \frac{13}{2} & 7 & -1 \end{pmatrix}^t \quad \therefore X = \begin{pmatrix} 2 & \frac{13}{2} \\ \frac{3}{2} & 7 \\ \frac{5}{2} & -1 \end{pmatrix}$$

Example 7

If $X + 2\,X^t = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}$ **, then find the matrix X**

Solution

$\because X + 2\,X^t = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}$ (1) (Take the transpose of both sides)

$\therefore (X + 2\,X^t)^t = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}^t$

Notice that

- $(A + B)^t = A^t + B^t$
- $(A^t)^t = A$

$\therefore X^t + 2X = \begin{pmatrix} 9 & 13 \\ 14 & 6 \end{pmatrix}$ (2)

Multiply (2) by – 2 :

$\therefore -2X^t - 4X = \begin{pmatrix} -18 & -26 \\ -28 & -12 \end{pmatrix}$ (3)

Add (1) and (3) :

$\therefore -3X = \begin{pmatrix} -9 & -12 \\ -15 & -6 \end{pmatrix}$

$\therefore X = \frac{-1}{3}\begin{pmatrix} -9 & -12 \\ -15 & -6 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$

TRY TO SOLVE

If $A = \begin{pmatrix} 2 & 4 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & -2 \\ 4 & \frac{1}{2} \end{pmatrix}$ **, then find the matrix X where :**

$3A - 2B = 2X - 3I$ where I is of the order 2×2

Remark

You can use the scientific calculator to add and subtract the matrices , and we will show that at the end of the unit.

Notice that

For any square matrix A :

$$A = \underbrace{\frac{1}{2}(A + A^t)}_{\text{Symmetric matrix}} + \underbrace{\frac{1}{2}(A - A^t)}_{\text{Skew symmetric matrix}}$$

Lesson **3**

Multiplying matrices

Introductory example

If the matrix A expresses the results of 20 matches for Al-Ahly team and Zamalek team in general league of football where :

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{l} \begin{array}{ccc} \text{Win} & \text{Drawn} & \text{Loss} \end{array} \\ \begin{pmatrix} 12 & 6 & 2 \\ 11 & 4 & 5 \end{pmatrix} \end{array} \begin{array}{l} \\ \longrightarrow \text{Al-Ahly} \\ \longrightarrow \text{Zamalek} \end{array}$$

and the matrix B expresses the number of points that the team gains in the cases of win , drawn and loss where :

$$B = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longrightarrow \text{Win} \\ \longrightarrow \text{Drawn} \\ \longrightarrow \text{Loss} \end{array}$$

, then the sum of points that Al-Ahly gained $= 12 \times 3 + 6 \times 1 + 2 \times 0 = 42$ points

, the sum of points that Zamalek gained $= 11 \times 3 + 4 \times 1 + 5 \times 0 = 37$ points and we can express the sum of points that each team gained by the matrix $C = \begin{pmatrix} 42 \\ 37 \end{pmatrix}$

We notice that : 42 is the sum of products of the elements of first row in the matrix A by the elements of the column in the matrix B , 37 is the sum of products of the elements of second row in the matrix A by the elements of the column in the matrix B

- The matrix C is the product of multiplying the matrix A × the matrix B

i.e. $C = A\,B = \begin{pmatrix} 12 & 6 & 2 \\ 11 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \times 3 + 6 \times 1 + 2 \times 0 \\ 11 \times 3 + 4 \times 1 + 5 \times 0 \end{pmatrix} = \begin{pmatrix} 42 \\ 37 \end{pmatrix}$

Multiplying matrices

If A is a matrix of the order $m \times \ell$, B is a matrix of the order $r \times n$, then :

- Their product $C = AB$ will be possible if and only if $\ell = r$

i.e. The number of columns of the matrix A = the number of rows of the matrix B

- The matrix $C = AB$ will be of the order $m \times n$

$$\underset{m \times \boxed{\ell \quad \text{Equal} \quad r} \times n}{A \qquad\qquad\qquad B} = \underset{m \times n}{(AB)}$$

- Each element c_{ij} in the matrix $C = AB$ equals the sum of products of elements of i^{th} row in the matrix A by the elements of j^{th} column in the matrix B , one by one corresponding to it.

To explain the concept of multiplying matrices :

For example :

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then A is a matrix of order 3×2 and B is a matrix of order 2×2 Since , the number of columns of matrix A = the number of rows of matrix B = 2 , then :

$$\underset{3 \times \boxed{2 \quad \text{Equal} \quad 2} \times 2}{A \qquad\qquad\qquad B} = \underset{3 \times 2}{(AB)}$$

i.e. The multiplying operation of the matrix A by the matrix B is possible and produces a matrix AB of order 3×2 and can be obtained as follow :

- Multiply each element from the first row in the matrix A by the corresponding element in the first column in the matrix B and adding up their products to get the element in (the first row and first column) in the matrix (AB) as follow :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$$

- Then multiply each element from the first row in the matrix A by the corresponding element in the second column in the matrix B and adding up their products to get the element in (the first row and the second column) in the matrix (AB) as follow :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \cdots & a_{11}b_{12} + a_{12}b_{22} \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$$

- and so on till we get all elements of the matrix AB as follow :

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} & a_{11}\,b_{12} + a_{12}\,b_{22} \\ a_{21}\,b_{11} + a_{22}\,b_{21} & a_{21}\,b_{12} + a_{22}\,b_{22} \\ a_{31}\,b_{11} + a_{32}\,b_{21} & a_{31}\,b_{12} + a_{32}\,b_{22} \end{pmatrix}$$

Notice that

The multiplying operation of matrix B by matrix A is not possible.

i.e. BA is not possible because the number of columns of matrix B $\neq$ the number of rows of matrix A

Example 1

Find AB if possible in each of the following :

1 $A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$

2 $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 & 3 \\ -1 & 2 & -4 \end{pmatrix}$

3 $A = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

Solution

1 $\because$ A is a matrix of order 3×2 and B is a matrix of order 2×2

$\therefore$ The number of columns of matrix A = the number of rows of matrix B

$\therefore$ AB is possible and of order 3×2

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} (2)(-1) + (1)(0) & (2)(2) + (1)(-3) \\ (3)(-1) + (-1)(0) & (3)(2) + (-1)(-3) \\ (0)(-1) + (4)(0) & (0)(2) + (4)(-3) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ -3 & 9 \\ 0 & -12 \end{pmatrix}$$

2 $\because$ A is a matrix of order 2×3 and B is a matrix of order 2×3

$\therefore$ The number of columns of matrix A $\neq$ the number of rows of matrix B

$\therefore$ AB is not possible.

3 $\because$ A is a matrix of order 1×3 and B is a matrix of order 3×1

$\therefore$ The number of columns of matrix A = the number of rows of matrix B = 3

$\therefore$ AB is possible and of order 1×1

$$\therefore AB = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} (2)(-2) + (-1)(1) + (3)(4) \end{pmatrix} = \begin{pmatrix} 7 \end{pmatrix}$$

Example 2

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 & 3 \\ -1 & 2 & -4 \end{pmatrix}$

, **find :** AB^t if possible.

Solution

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix}$ of order 2×3 , $B^t = \begin{pmatrix} 1 & -1 \\ 5 & 2 \\ 3 & -4 \end{pmatrix}$ of order 3×2

$\because$ The number of columns of matrix A = the number of rows of matrix B^t

$\therefore AB^t$ is possible and of order 2×2

$$\therefore AB^t = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 20 & -9 \\ -5 & 7 \end{pmatrix}$$

TRY TO SOLVE

If $A = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 0 \\ 4 & -1 \\ 2 & 1 \end{pmatrix}$, **then find if possible :** AB , AB^t , $A^t B$

Properties of multiplying matrices

If A , B and C are three matrices , I is the identity matrix, then the following properties are satisfied.

1 Associative property :

$\boxed{(AB)C = A(BC)}$ where multiplying operations are possible.

For example :

If $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 0 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

, then $AB = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & -4 \\ 2 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -23 \\ 5 & -1 & 24 \end{pmatrix}$

$\therefore (AB)C = \begin{pmatrix} 0 & 2 & -23 \\ 5 & -1 & 24 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 19 \\ -17 \end{pmatrix}$

, $BC = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

$\therefore A(BC) = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 19 \\ -17 \end{pmatrix}$

$\therefore (AB)C = A(BC)$

2 The existence of multiplicative neutral (identity) property :

The identity matrix I is the multiplicative neutral matrix.

i.e. $\boxed{AI = IA = A}$ where A is a square matrix of the same order of I

For example :

$$\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$$

3 Distributing multiplication of matrices on addition property :

$\boxed{\begin{array}{l} A(B + C) = AB + AC \\ ,(A + B)C = AC + BC \end{array}}$ where multiplying and adding operations are possible.

For example :

If $A = \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix}$

, then $B + C = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\therefore A(B + C) = \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & -3 \end{pmatrix}$ (1)

, $\because AB + AC = \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -9 \\ -9 & 15 \end{pmatrix} + \begin{pmatrix} -4 & 8 \\ 6 & -18 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & -3 \end{pmatrix}$ (2)

$\therefore$ From (1) and (2) , we deduce that : $A(B + C) = AB + AC$

Remark

If A and B are two matrices whose multiplying operation is possible in any form i.e. AB is possible and BA is possible too , then it is not necessary that : $AB = BA$ that means that : the multiplying operation is not commutative.

For example :

If $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

, then :

1 $AB = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} -23 & -14 \\ -9 & -4 \end{pmatrix}$

, $BA = \begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -6 & 5 \\ 32 & -21 \end{pmatrix}$

Notice that

It is possible to multiply any two square matrices of the same order.

i.e. $AB \neq BA$

2 $AC = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & 6 \\ -4 & 2 \end{pmatrix}$

, $CA = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -8 & 6 \\ -4 & 2 \end{pmatrix}$

i.e. $AC = CA$

Example 3

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, **find :** A^2, A^3

Solution

$$A^2 = A \times A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ -4 & -1 \end{pmatrix}$$

$$, A^3 = A^2 \times A = \begin{pmatrix} 7 & 8 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 22 \\ -11 & -9 \end{pmatrix}$$

Notice that

If A is not a square matrix, then A^2 is not possible.

Example 4

If $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$, **then prove that :** $A^2 - 2A - 3I^2 = O$

Solution

$$A^2 - 2A - 3I^2 = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} - 2\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

Notice that

- $I^2 = I$
- $I^n = I$ for every $n \in \mathbb{Z}^+$

TRY TO SOLVE

If $A = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$, **then prove that :** $A^2 - 5A + 2I = O$

Critical thinking

1 If A and B are two matrices, $AB = O$

Does it mean that $A = O$ or $B = O$ always ?

Answer : No

Explanation of the answer :

Let $A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$, then

$$AB = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$, \because A \neq O, B \neq O$

i.e. If $AB = O$, it is not always that : $A = O$ or $B = O$

2 If A is a square matrix and $A^2 = I$. Does that mean for sure $A = I$?

Answer : No.

let $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

i.e. If $A^2 = I$ so it is not necessary that $A = I$

3 If A and B are two matrices and $A \times B = A$. Does that mean for sure $B = I$?

Answer : No.

let $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$, then $A \times B = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \times \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = A$

i.e. If $A \times B = A$ so it is not necessary that $B = I$

Transpose of the product of two matrices

If A and B are two matrices and AB is possible , then $(AB)^t = B^t A^t$

Generally ,

$(ABC \text{ .... } E)^t = E^t \text{ .... } C^t B^t A^t$ where multiplying operations are possible.

Example 5

If $A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 1 \\ -3 & 9 \\ 2 & -8 \end{pmatrix}$, **check that :** $(AB)^t = B^t A^t$

Solution

$\because AB = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 5 & 7 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ -3 & 9 \\ 2 & -8 \end{pmatrix} = \begin{pmatrix} 23 & -39 \\ 17 & -8 \end{pmatrix}$

$\therefore (AB)^t = \begin{pmatrix} 23 & 17 \\ -39 & -8 \end{pmatrix}$ (1)

, $\because B^t = \begin{pmatrix} 6 & -3 & 2 \\ 1 & 9 & -8 \end{pmatrix}$, $A^t = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 4 & 7 \end{pmatrix}$

$\therefore B^t A^t = \begin{pmatrix} 6 & -3 & 2 \\ 1 & 9 & -8 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 23 & 17 \\ -39 & -8 \end{pmatrix}$ (2)

From (1) and (2) : $\therefore (AB)^t = B^t A^t$

Example 6

If $A = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 & 6 \\ 5 & -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -3 \\ -5 & 2 \\ 3 & 4 \end{pmatrix}$,

find the matrix X that satisfies the relation : $15\,X^t = A^2 + (BC)^t$

Solution

$\because A^2 = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 7 & -7 \\ -21 & 28 \end{pmatrix}$

, $BC = \begin{pmatrix} -2 & 3 & 6 \\ 5 & -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -5 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 36 \\ 52 & -13 \end{pmatrix}$ $\quad \therefore (BC)^t = \begin{pmatrix} 1 & 52 \\ 36 & -13 \end{pmatrix}$

$\therefore 15\,X^t = \begin{pmatrix} 7 & -7 \\ -21 & 28 \end{pmatrix} + \begin{pmatrix} 1 & 52 \\ 36 & -13 \end{pmatrix} = \begin{pmatrix} 8 & 45 \\ 15 & 15 \end{pmatrix}$

$\therefore X^t = \begin{pmatrix} \frac{8}{15} & 3 \\ 1 & 1 \end{pmatrix}$ $\qquad \therefore X = \begin{pmatrix} \frac{8}{15} & 1 \\ 3 & 1 \end{pmatrix}$

Example 7

Find the values of a , b and c if :

$$\begin{pmatrix} 1 & a & 2 \\ 0 & 2 & 4 \\ 5 & -1 & b \end{pmatrix}\begin{pmatrix} -1 & 0 & 2 \\ 7 & 6 & 4 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} -19 & -6 & 4 \\ 6 & c & 28 \\ -24 & 12 & 36 \end{pmatrix}$$

Solution

We can find a , b and c without carrying out a complete multiplication operation, but we will just :

- Multiply the elements of the first row of the first matrix by the elements of the first column of the second matrix

$\therefore 1 \times -1 + a \times 7 + 2 \times -2 = -19 \qquad \therefore a = -2$

- Multiply the elements of the third row of the first matrix by the elements of the first column of the second matrix.

$\therefore 5 \times -1 - 1 \times 7 + b \times -2 = -24 \qquad \therefore b = 6$

- Multiply the elements of the second row of the first matrix by the elements of the second column of the second matrix.

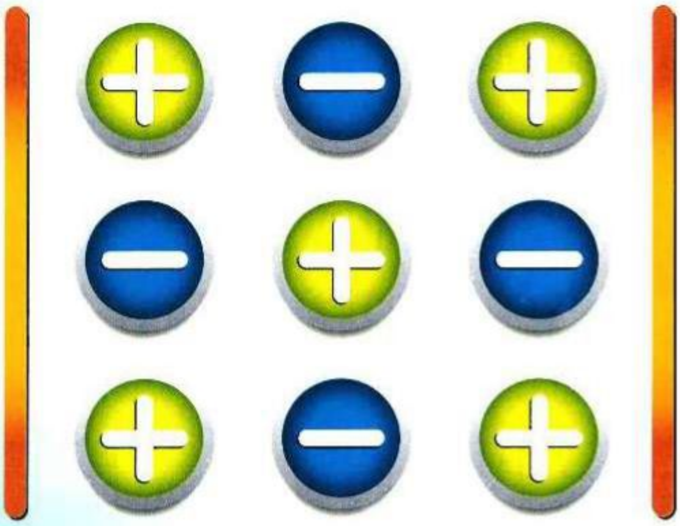
$\therefore 0 \times 0 + 2 \times 6 + 4 \times 3 = c \qquad \therefore c = 24$

Remark

We can use the scientific calculator for multiplying matrices and we will represent it at the end of the unit.

Lesson 4

Determinants



The second order determinant

If A is a square matrix of order 2×2 where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of the matrix A is denoted by the symbol $|A|$ and is called determinant of the second order and it is the number defined as follows :

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

i.e. The value of the determinant of the second order equals the product of the two elements of the principal diagonal minus the product of the two elements of the other diagonal.

Example 1

Find the value of each of the following determinants :

1 $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix}$

2 $\begin{vmatrix} 4 & -7 \\ 2 & 6 \end{vmatrix}$

3 $\begin{vmatrix} 6 & 3 \\ -8 & -4 \end{vmatrix}$

4 $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

Solution

1 $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2 \times 8 - 3 \times 5 = 16 - 15 = 1$

2 $\begin{vmatrix} 4 & -7 \\ 2 & 6 \end{vmatrix} = 4 \times 6 - 2 \times (-7) = 24 + 14 = 38$

3 $\begin{vmatrix} 6 & 3 \\ -8 & -4 \end{vmatrix} = 6 \times (-4) - (-8) \times 3 = -24 + 24 = 0$

4 $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = \sin\theta \times \sin\theta - (-\cos\theta) \times \cos\theta = \sin^2\theta + \cos^2\theta = 1$

TRY TO SOLVE

Find the value of each of the following determinants :

1 $\begin{vmatrix} 3 & 7 \\ 0 & -4 \end{vmatrix}$ **2** $\begin{vmatrix} -1 & 3 \\ -5 & -2 \end{vmatrix}$

Example 2

Find the value of x which satisfies each of the following equations :

1 $\begin{vmatrix} x^2-4 & 1 \\ 0 & 1 \end{vmatrix} = 0$ **2** $\begin{vmatrix} x+2 & 3 \\ -2 & x-2 \end{vmatrix} = 1$

Solution

1 $\because \begin{vmatrix} x^2-4 & 1 \\ 0 & 1 \end{vmatrix} = (x^2-4) \times 1 - 0 \times 1 = x^2 - 4$

$\therefore x^2 - 4 = 0$ $\therefore x^2 = 4$ $\therefore x = \pm\sqrt{4} = \pm 2$

2 $\because \begin{vmatrix} x+2 & 3 \\ -2 & x-2 \end{vmatrix} = (x+2)(x-2) - (-2) \times 3 = x^2 - 4 + 6 = x^2 + 2$

$\therefore x^2 + 2 = 1$ $\therefore x^2 = -1$ $\therefore x = \pm\sqrt{-1}$

$\therefore x = i$ or $x = -i$ (where $i^2 = -1$)

TRY TO SOLVE

Find the value of x which satisfies the equation :

$\begin{vmatrix} 2x & -2 \\ 4 & 1 \end{vmatrix} = 6$

The third order determinant

If A is a square matrix of order 3×3 where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

, then the determinant of the matrix A is denoted by the symbol $|A|$ and is called

determinant of the third order where $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

And before knowing how to expand the third order determinant , we will recognize the "minor determinant" corresponding to any element of the matrix A , and how to determine its sign.

For every element of the matrix A there exists a minor determinant which we can get by elemenating the row and the column intersected at this element.

For example : We can get the minor determinant corresponding to each element of the first row as follows :

To get the minor determinant corresponding to a_{11} which is denoted by $|a_{11}|$ ➡ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ ➡ $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

To get the minor determinant corresponding to a_{12} which is denoted by $|a_{12}|$ ➡ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ ➡ $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

To get the minor determinant corresponding to a_{13} which is denoted by $|a_{13}|$ ➡ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ ➡ $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- To determine the sign of the minor determinant of any element of a matrix , we add the two orders of the row and the column intersected at this element.

If the sum of the two orders is : • **even :** then the sign is positive.

• **odd :** then the sign is negative.

For example :

- The sign of $|a_{11}|$ is positive because $1 + 1 = 2$ (even)
- The sign of $|a_{12}|$ is negative because $1 + 2 = 3$ (odd)
- The sign of $|a_{13}|$ is positive because $1 + 3 = 4$ (even)

• Hence , we can write the rule of signs of the minor determinant as in the opposite figure :

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Note that :

The sign of the minor determinant corresponding to the element a_{ij} is determined by the rule : $(-1)^{i+j}$

Expanding the third order determinant

It is possible to expand the third order determinant in terms of the elements of any row or column and its minor determinants , taking into account the rule of signs.

For example : If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then :

$$* \ |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$ (Using the elements of the first row)

$$* \ |A| = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$ (Using the elements of the second column)

Example 3

Find the value of the determinant : $\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix}$

Solution

Using the elements of the first row , we find that :

$$\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 2\,(0 \times (-3) - 1 \times 4) - 3\,(-2 \times (-3) - 1 \times 4) - (-2 \times 1 - 1 \times 0)$$

$$= 2\,(0 - 4) - 3\,(6 - 4) - (-2 - 0)$$

$$= 2 \times (-4) - 3 \times 2 + 2 = -12$$

Remark

It is possible to expand the determinant using any row or column as mentioned before , and we will expand it again using the elements of the second column taking into account the rule of signs.

$$\begin{vmatrix} 2 & 3 & -1 \\ -2 & 0 & 4 \\ 1 & 1 & -3 \end{vmatrix} = -3\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ -2 & 4 \end{vmatrix}$$

$$= -3\left(-2 \times (-3) - 1 \times 4\right) + 0 - \left(2 \times 4 - (-2) \times (-1)\right)$$

$$= -3\,(6-4) - (8-2) = -3 \times 2 - 6 = -12$$

Which is the same result we get before (try to use any other row or column)

Example 4

Find the value of the determinant : $\begin{vmatrix} 4 & -1 & 3 \\ 0 & 5 & -2 \\ 0 & -3 & -1 \end{vmatrix}$

Solution

It is preferable to expand this determinant in terms of the elements of the first column because of the existence of the greatest number of zeroes

$\therefore$ The value of the determinant $= 4\begin{vmatrix} 5 & -2 \\ -3 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ -3 & -1 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix}$

$$= 4\left(5 \times (-1) - (-3) \times (-2)\right) - 0 + 0$$

$$= 4\,(-5-6) = 4 \times (-11) = -44$$

TRY TO SOLVE

Find the value of the determinant : $\begin{vmatrix} 3 & 0 & -5 \\ -2 & 4 & 1 \\ 7 & -3 & 6 \end{vmatrix}$

Some Properties of determinants

1 In any determinant if the rows are replaced by the columns in the same order , the value of the determinant is unchanged.

In other words :

The value of the determinant of a square matrix equals the value of the determinant of the transpose of this matrix.

So , if A is a square matrix , then $|A| = |A^t|$

For example : $|A| = \begin{vmatrix} 1 & -3 & 5 \\ 4 & 0 & 7 \\ 8 & 14 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 8 \\ -3 & 0 & 14 \\ 5 & 7 & -3 \end{vmatrix}$

It can be proved by expanding both determinants as follow :

$$\begin{vmatrix} 1 & -3 & 5 \\ 4 & 0 & 7 \\ 8 & 14 & -3 \end{vmatrix} = 1 \times -98 + 3 \times (-68) + 5 \times 56$$

$= -22$ (by using the elements of the first row)

$$\begin{vmatrix} 1 & 4 & 8 \\ -3 & 0 & 14 \\ 5 & 7 & -3 \end{vmatrix} = 1 \times -98 - 4 \times (-61) + 8 \times (-21) = -22$$

(by using the elements of the first row)

2 The value of the determinant vanishes in each of the following two cases.

1 If all the elements of any row (column) in any determinant equal zero :

For example : The value of the determinant $\begin{vmatrix} 2 & 1 & 0 \\ 4 & -3 & 0 \\ -1 & 5 & 0 \end{vmatrix} = 0$ because all elements of the third column (C_3) are zeroes

2 If all the corresponding elements of any two rows (columns) in any determinant are equal :

For example : $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -3 & 5 \\ 2 & 1 & 3 \end{vmatrix} = 0$, because the corresponding elements in first and third row are equal , (written as $R_1 = R_3$)

3 If there is a common factor in all elements of any row (column) in a determinant , then this factor can be taken outside the determinant.

For example : If $|A| = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 7 & 5 & 3 \end{vmatrix}$ take 2 as a common factor from the elements of the second row (R_2)

$\therefore |A| = 2 \begin{vmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 7 & 5 & 3 \end{vmatrix}$ and notice that , elements of first and second row are equal ($R_2 = R_1$) $\quad \therefore |A| = 2 \times \text{zero} = \text{zero}$

Remarks

1 From property 3 , when multiplying a determinant by a non zero real number k , then multiply this number by elements of only one row (column)

For example : $k \begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & kb & c \\ d & ke & f \\ x & ky & z \end{vmatrix} = \cdots\cdots$ and so on

2 The value of a determinant vanished if the elements of any row (column) is multiple of elements of another row (column) in the determinant.

For example : The value of the determinant $\begin{vmatrix} 3 & 3 & 1 \\ -6 & -1 & -2 \\ 15 & 6 & 5 \end{vmatrix}$ = zero

because each element in the 1^{st} column is 3 times the corresponding element in 3^{rd} column (briefly written $C_1 = 3\, C_3$)

4 In any determinant , if the positions of two rows (columns) are interchanged. The value of the resulting determinant equals the additive inverse of the value of the original detrminant.

In other words : If the position of two rows (columns) are interchanged , then the resulting determinant = – the original determinant.

For example :

If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = 10$, if the first and second rows are interchanged (R_1 , R_2)

then $\begin{vmatrix} d & e & f \\ a & b & c \\ x & y & z \end{vmatrix} = -10$ **i.e.** $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ x & y & z \end{vmatrix}$

and that can be proved by expanding both determinants.

The determinant of the triangular matrix

The triangular matrix

It is a matrix in which all its elements above or below the principal diagonal are zeros as :

$$\begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 4 & 2 & 7 \end{pmatrix}$$

The value of the determinant of the triangular matrix equals the product of the elements of its principal diagonal.

i.e. $\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}\, a_{22}$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\, a_{22}\, a_{33}$

Proof : $\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}\, a_{22} - 0 \times a_{12} = a_{11}\, a_{22} - 0 = a_{11}\, a_{22}$

$$,\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{12} & a_{13} \\ 0 & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$ (Using the elements of the first column)

$$= a_{11}\,(a_{22}\,a_{33} - a_{23} \times 0) = a_{11}\,a_{22}\,a_{33}$$

, then $\begin{vmatrix} 5 & 0 \\ -3 & 2 \end{vmatrix} = 10$, $\begin{vmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 4 & 2 & 7 \end{vmatrix} = 2 \times (-3) \times 7 = -42$

TRY TO SOLVE

Find the value of the determinant : $\begin{vmatrix} -1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{vmatrix}$

Example 5

Solve the equation : $\begin{vmatrix} x & 0 & 1 \\ 8 & 1-x & -x \\ x & -1 & 1+x \end{vmatrix} = 0$

Solution

Expanding the determinant :

$$\therefore x\begin{vmatrix} 1-x & -x \\ -1 & 1+x \end{vmatrix} - 0\begin{vmatrix} 8 & -x \\ x & 1+x \end{vmatrix} + 1\begin{vmatrix} 8 & 1-x \\ x & -1 \end{vmatrix} = 0$$

$$\therefore x\Big((1-x)(1+x) - (-x) \times (-1)\Big) - 0 + 1\Big(8 \times (-1) - (1-x) \times x\Big) = 0$$

$\therefore x(1 - x^2 - x) + 1(-8 - x + x^2) = 0$ $\qquad$ $\therefore x - x^3 - x^2 - 8 - x + x^2 = 0$

$\therefore -x^3 - 8 = 0$ $\qquad$ $\therefore x^3 = -8$

$\therefore\ x = -2$

TRY TO SOLVE

Solve the equation : $\begin{vmatrix} x & 2 & -2 \\ 2 & x & -2 \\ -2 & 2 & x \end{vmatrix} = 0$

Example 6

If A is a matrix of order 2×2 and $|A| = 7$, find $|3A|$

Solution

Let $A = \begin{pmatrix} x & y \\ z & \ell \end{pmatrix}$ $\therefore |A| = x\ell - yz = 7$ (1)

, $3A = 3\begin{pmatrix} x & y \\ z & \ell \end{pmatrix} = \begin{pmatrix} 3x & 3y \\ 3z & 3\ell \end{pmatrix}$

$\therefore |3A| = \begin{vmatrix} 3x & 3y \\ 3z & 3\ell \end{vmatrix} = 9x\ell - 9yz = 9(x\ell - yz)$ (2)

from (1) , (2) : $\therefore |3A| = 9 \times 7 = 63$

from the previous example we can conclude that :

Remarks

1. If A is a matrix of order $n \times n$, $K \in \mathbb{R}$, then $|KA| = K^n |A|$

For example :

- If A is a matrix of order 2×2 and $|A| = 3$
  , then $|5A| = 5^2 \times |A| = 25 \times 3 = 75$
- If A is a matrix of order 3×3 and $|A| = 5$
  , then $|2A| = 2^3 \times |A| = 8 \times 5 = 40$

2. If A and B are two square matrices such that AB exists , then $|AB| = |A| \times |B|$

Finding the area of a triangle by using determinants

We can use determinants to find the area of a triangle in terms of the coordinates of its vertices as follows :

If XYZ is a triangle where X (a , b) , Y (c , d) , Z (e , f)

, then the area of Δ XYZ is $|A|$

$$\text{Where } A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

x – coordinates of the triangle vertices

y – coordinates of the triangle vertices

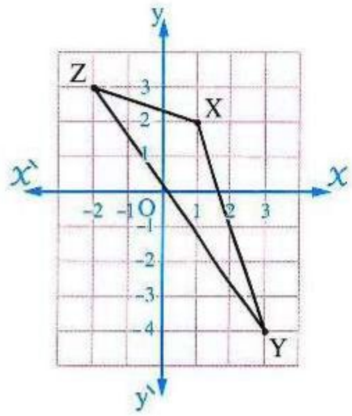
Remember that

$|A|$ means the absolute value of A (i.e. only its positive value)

And we will represent the proof of the previous law at the end of this lesson as an enrich activity.

Example 7

Find using determinants the area of the opposite triangle whose vertices are X (1 , 2) , Y (3 , – 4) , Z (– 2 , 3)



Solution

$$\because A = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & 1 \\ -2 & 3 & 1 \end{vmatrix}$$

By using the elements of the third column

$$\therefore A = \frac{1}{2}\left[\begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left((9-8)-(3+4)+(-4-6)\right)$$

$$= \frac{1}{2}(1-7-10) = -8$$

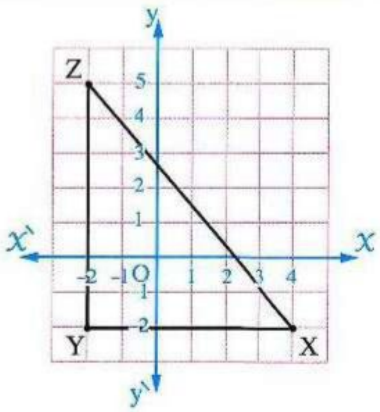
$\therefore$ The area of Δ XYZ = | A | = | – 8 | = 8 square units.

Note that we used the elements of the third row to expand the determinant because it is easily for performing mathematical operations because of the existence of ones.

TRY TO SOLVE

In the opposite figure :

XYZ is a triangle where X = (4 , –2) , Y = (–2 , –2) , Z = (–2 , 5) find using determinants the area of Δ XYZ , and check your answer using the rule of calculating the triangle area.



Remark

To prove that the three points X (a , b) , Y (c , d) , Z (e , f) are collinear by using determinants , then we prove that : $\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix} = 0$

Example 8

Prove using determinants that the points (–2 , 4) , (3 , 0) , (8 , –4) are collinear.

Solution

$$\because \begin{vmatrix} -2 & 4 & 1 \\ 3 & 0 & 1 \\ 8 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 8 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 4 \\ 8 & -4 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix}$$

$$= (-12 - 0) - (8 - 32) + (0 - 12)$$

$$= -12 + 24 - 12 = 0$$

$\therefore$ The points (– 2 , 4) , (3 , 0) , (8 , – 4) are collinear.

TRY TO SOLVE

Prove using determinants that the points (4 , 4) , (2 , 1) , (–2 , – 5) are collinear.

Solving a system of linear equations by Cramer's rule

First Solving a system of linear equations in two variables

- Solving a system of linear equations in two variables means to find the values of the two variables satisfying the two equations together.
- If we have a system of linear equations in two variables as follows : $\begin{matrix} a\,x + b\,y = m \\ c\,x + d\,y = n \end{matrix}$, then to solve this system we do the following :

1 Find the values of three determinants , after putting the two equations in the previous form , and these determinants are :

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	• Is called the determinant of the matrix of coefficients and denoted by the symbol Δ (is read as delta) • We get it by putting the two coefficients of x in the two equations in the first column , and the two coefficients of y in the two equations in the second column.
$\begin{vmatrix} m & b \\ n & d \end{vmatrix}$	• Is called the determinant of the variable x and denoted by the symbol Δ_x (is read as delta x) • We get it from the determinant Δ by changing the elements of the first column (coefficients of x) by the constants m and n
$\begin{vmatrix} a & m \\ c & n \end{vmatrix}$	• Is called the determinant of the variable y and is denoted by the symbol Δ_y (is read as delta y) • We get it from the determinant Δ by changing the elements of the second column (coefficients of y) by the constants m and n

2 Find the values of x and y as follows (where $\Delta \neq 0$) :

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{md - nb}{ad - cb}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{an - cm}{ad - cb}$$

* **Note that :** If $\Delta \neq 0$, then the system has a unique solution while if $\Delta = 0$, then the system either has an infinite number of solutions or has no solution.

The following example shows the previous steps.

Example 9

Solve the system of the following equations using Cramer's rule :

$6x - 5y = -23$, $3x + 3y = 16$

Solution

$$\Delta = \begin{vmatrix} 6 & -5 \\ 3 & 3 \end{vmatrix} = 6 \times 3 - 3 \times (-5) = 18 + 15 = 33$$

$$, \Delta_x = \begin{vmatrix} -23 & -5 \\ 16 & 3 \end{vmatrix} = -23 \times 3 - 16 \times (-5) = -69 + 80 = 11$$

$$, \Delta_y = \begin{vmatrix} 6 & -23 \\ 3 & 16 \end{vmatrix} = 6 \times 16 - 3 \times (-23) = 96 + 69 = 165$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{11}{33} = \frac{1}{3}$$

$$, y = \frac{\Delta_y}{\Delta} = \frac{165}{33} = 5$$

and the S.S. $= \left\{\left(\frac{1}{3} , 5\right)\right\}$

Remark

You can check your answer by substituting the values of x and y in the two equations.

TRY TO SOLVE

Solve the following two equations using Cramer's rule :

$4x + 3y = -4$, $3x - y = -3$

Second Solving a system of linear equations in three variables

If we have a system of linear equations in three variables as follows :

1 $a_1 X + b_1 y + c_1 z = m$ **2** $a_2 X + b_2 y + c_2 z = n$ **3** $a_3 X + b_3 y + c_3 z = k$

, then similarly as we did in case of system of linear equations in two variables :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{determinant of the coefficients}$$

$$\Delta_X = \begin{vmatrix} m & b_1 & c_1 \\ n & b_2 & c_2 \\ k & b_3 & c_3 \end{vmatrix} = \text{determinant of the variable } X$$

and we get it by changing the elements of the first column (coefficients of X) by the constants m , n , k

$$\Delta_y = \begin{vmatrix} a_1 & m & c_1 \\ a_2 & n & c_2 \\ a_3 & k & c_3 \end{vmatrix} = \text{determinant of the variable y}$$

and we get it by changing the elements of the second column (coefficients of y) by the constants m , n , k

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & m \\ a_2 & b_2 & n \\ a_3 & b_3 & k \end{vmatrix} = \text{determinant of the variable z}$$

and we get it by changing the elements of the third column (coefficients of z) by the constants m , n , k

Let $\Delta \neq 0$, then $X = \dfrac{\Delta_X}{\Delta}$, $y = \dfrac{\Delta_y}{\Delta}$, $z = \dfrac{\Delta_z}{\Delta}$

The following example shows the previous steps.

Example 10

Solve the system of the following equations using Cramer's rule :

$3y + 2X = z + 1$, $3X + 2z = 8 - 5y$, $3z - 1 = X - 2y$

Solution

1 Put the equations system in the form $aX + by + cz = m$ as follows :

$2X + 3y - z = 1$, $3X + 5y + 2z = 8$, $X - 2y - 3z = -1$

2 Find Δ , Δ_x , Δ_y , Δ_z as follows :

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix} = 2\,(-15+4) - 3\,(-9-2) + (-1)\,(-6-5)$$

$$= -22 + 33 + 11 = 22$$

$$, \Delta_x = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix} = 1\,(-15+4) - 3\,(-24+2) + (-1)\,(-16+5)$$

$$= -11 + 66 + 11 = 66$$

$$, \Delta_y = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix} = 2\,(-24+2) - 1\,(-9-2) + (-1)\,(-3-8)$$

$$= -44 + 11 + 11 = -22$$

$$, \Delta_z = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix} = 2\,(-5+16) - 3\,(-3-8) + 1\,(-6-5)$$

$$= 22 + 33 - 11 = 44$$

3 Find the variables x , y , z as follows :

$$x = \frac{\Delta_x}{\Delta} = \frac{66}{22} = 3 \quad , y = \frac{\Delta_y}{\Delta} = \frac{-22}{22} = -1 \quad , z = \frac{\Delta_z}{\Delta} = \frac{44}{22} = 2$$

$\therefore$ The S.S. $= \{(3 , -1 , 2)\}$

Remarks

- You can check your answer by substituting the three variables in each equation.
- (3 , –1 , 2) is called ordered triple.

TRY TO SOLVE

Solve the system of the following equations using Cramer's rule :

$2x + y - z = 3 \quad , \quad x + y = 1 - z \quad , \quad x = 2y + 3z + 4$

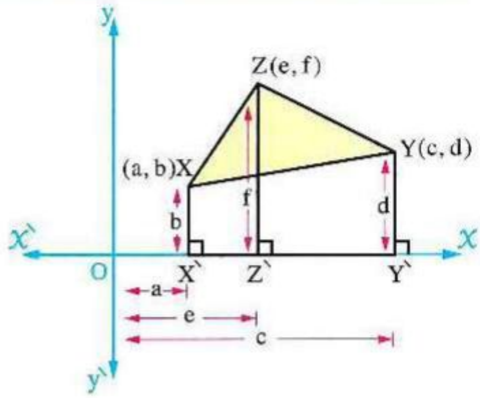
Remark

You can use the scientific calculator to find the value of the determinant and we will represent it at the end of the unit.

Remark

Scientific calculator can be used to calculate the value of the determinant and we will show that at the end of the unit.

Activity A method to prove the law of finding the area of the triangle using the determinants



Let XYZ be a triangle where

X (a , b) , Y (c , d) , Z (e , f) , then

The area of Δ XYZ = the area of the trapezium $X\grave{X}\grave{Z}Z$

+ the area of the trapezium $Z\grave{Z}\grave{Y}Y$

− the area of the trapezium $X\grave{X}\grave{Y}Y$

$$= \frac{b+f}{2}(e-a) + \frac{f+d}{2}(c-e) - \frac{b+d}{2}(c-a)$$

$$= \frac{1}{2}\left[(b+f)(e-a) + (f+d)(c-e) - (b+d)(c-a)\right]$$

$$= \frac{1}{2}\left[be - \cancel{ba} + \cancel{fe} - fa + fc - \cancel{fe} + \cancel{dc} - de - bc + \cancel{ba} - \cancel{dc} + da\right]$$

$$= \frac{1}{2}\left[be - fa + fc - de - bc + da\right] \qquad (1)$$

and by expanding the determinant : $\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ using the elements of the third column , we find that :

$$\text{The determinant} = \frac{1}{2}\left[\begin{vmatrix} c & d \\ e & f \end{vmatrix} - \begin{vmatrix} a & b \\ e & f \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix}\right]$$

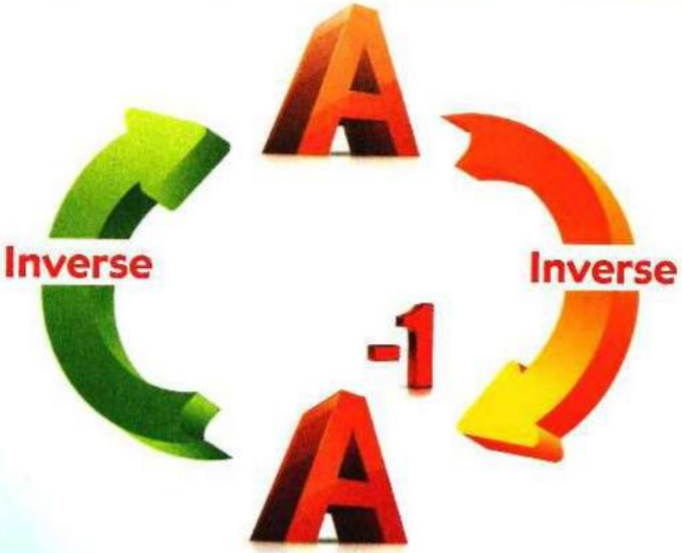
$$= \frac{1}{2}\left[cf - ed - af + eb + ad - bc\right] \qquad (2)$$

by comparing the result which we get in (1) and the result which we get in (2) , we find that :

The area of Δ XYZ = $\frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ (in condition of taking the absolute value of the result)

Lesson **5**

Multiplicative inverse of a matrix



If A and B are two square matrices and each of them is of order 2×2 and $AB = BA = I$ where I is the unit matrix of order 2×2 , then each of the two matrices A and B is the multiplicative inverse of the other.

For example :

If $A = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{-3}{2} & 2 \end{pmatrix}$

, then $AB = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{-3}{2} & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

, $BA = \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{-3}{2} & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

i.e. $AB = BA = I$

$\therefore$ Each of the two matrices A and B is the multiplicative inverse of the other.

Remark

If the matrix $A = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix}$ and the matrix $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$

then A is not the multiplicative inverse of B although $AB = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, that is because **the matrices A and B are not square matrices**.

Unit 1

How to find the multiplicative inverse of a 2 × 2 matrix ?

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the multiplicative inverse of the matrix A which is denoted by the symbol A^{-1} is defined (existed) when the determinant of $A = \Delta \neq 0$, then

$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $AA^{-1} = A^{-1}A = I$

Example 1

Find the multiplicative inverse if it existed of each of the following matrices :

1 $A = \begin{pmatrix} -2 & 2 \\ 3 & -4 \end{pmatrix}$

2 $B = \begin{pmatrix} \frac{1}{2} & 2 \\ 3 & 12 \end{pmatrix}$

Solution

1 $\because \Delta = \begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix} = (-2)(-4) - (3)(2) = 2$

$\therefore \Delta \neq 0$ $\quad\therefore$ The matrix A has a multiplicative inverse

$\therefore A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & -2 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ \frac{-3}{2} & -1 \end{pmatrix}$

2 $\because \Delta = \begin{vmatrix} \frac{1}{2} & 2 \\ 3 & 12 \end{vmatrix} = \left(\frac{1}{2}\right)(12) - (2)(3) = 0$

$\therefore B^{-1}$ is not defined (not existed)

TRY TO SOLVE

Find the multiplicative inverse if it possible of the matrix : $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

Example 2

Find the real values of x which make the matrix A has a multiplicative inverse in each of the following :

1 $A = \begin{pmatrix} x & 3 \\ 12 & x \end{pmatrix}$

2 $A = \begin{pmatrix} x-1 & 4 \\ 3 & x-2 \end{pmatrix}$

Solution

1 The matrix A has no multiplicative inverse when $|A| = 0$

i.e. $\begin{vmatrix} x & 3 \\ 12 & x \end{vmatrix} = 0$ $\therefore x^2 - 36 = 0$ $\therefore x = \pm 6$

$\therefore$ The matrix A has no multiplicative inverse when $x = \pm 6$

$\therefore$ The matrix A has a multiplicative inverse when $x \in \mathbb{R} - \{-6, 6\}$

2 The matrix A has no multiplicative inverse when $|A| = 0$

i.e. $\begin{vmatrix} x-1 & 4 \\ 3 & x-2 \end{vmatrix} = 0$ $\therefore (x-1)(x-2) - 12 = 0$

$\therefore x^2 - 3x + 2 - 12 = 0$ $\therefore x^2 - 3x - 10 = 0$

$\therefore (x-5)(x+2) = 0$ $\therefore x = 5$ or $x = -2$

$\therefore$ The matrix A has no multiplicative inverse when $x = 5$ or $x = -2$

$\therefore$ The matrix A has a multiplicative inverse when $x \in \mathbb{R} - \{5, -2\}$

TRY TO SOLVE

Find the real values of x which make the matrix : $A = \begin{pmatrix} x-1 & 4 \\ 2 & x+1 \end{pmatrix}$ has a multiplicative inverse.

Example 3

If $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$ **, then prove that :**

1 $(A^{-1})^{-1} = A$

2 $(AB)^{-1} = B^{-1} A^{-1}$

Solution

1 $\because |A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (1)(4) - (3)(2) = -2$ $\therefore \Delta \neq 0$

$\therefore A^{-1}$ is defined (existed)

, $\because A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$

$, \because |A^{-1}| = \begin{vmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{vmatrix} = (-2)\left(\frac{-1}{2}\right) - \left(\frac{3}{2}\right)(1) = \frac{-1}{2} \neq 0$

$\therefore (A^{-1})^{-1}$ is defined (existed)

$$\therefore (A^{-1})^{-1} = \frac{1}{\frac{-1}{2}} \begin{pmatrix} \frac{-1}{2} & \frac{-3}{2} \\ -1 & -2 \end{pmatrix} = -2 \begin{pmatrix} \frac{-1}{2} & \frac{-3}{2} \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = A$$

2 $\because AB = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ -8 & 2 \end{pmatrix}$

$\therefore |AB| = \begin{vmatrix} -7 & 2 \\ -8 & 2 \end{vmatrix} = (-7)(2) - (2)(-8) = 2 \neq 0$

$\therefore (AB)^{-1}$ is existed. $\qquad \therefore (AB)^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ 8 & -7 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & \frac{-7}{2} \end{pmatrix}$ (1)

$, |B| = \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} = (2)(1) - (-1)(-3) = -1$ $\qquad \therefore B^{-1}$ is existed.

$\therefore B^{-1} = \frac{-1}{1}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$ $\qquad , \because A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$

$\therefore B^{-1}A^{-1} = \begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & \frac{-7}{2} \end{pmatrix}$ (2)

From (1) and (2) , we get that : $(AB)^{-1} = B^{-1}A^{-1}$

TRY TO SOLVE

Using the two matrices A and B in the previous example , prove that : $(A^{-1}B)^{-1} = B^{-1}A$

Remark

If A is a square matrix of order 2×2 where $|A| \neq 0$, C is another matrix and :

1 $AX = C$ then $X = A^{-1}C$

By multiplying the two sides of the equation by A^{-1}

$\therefore A^{-1}AX = A^{-1}C \quad \therefore IX = A^{-1}C \quad \therefore X = A^{-1}C$

2 $XA = C$ then $X = CA^{-1}$

Notice that :

* $A^{-1}A = I$, $IX = X$
* $|A^{-1}| = \frac{1}{|A|}$

Example 4

Find the matrix X which satisfies that : $\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \times X = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$, $C = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\therefore$ The equation is $AX = C$ $\qquad \therefore X = A^{-1} C$

, $\because |A| = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = (2)(0) - (-1)(3) = 3 \neq 0$

$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}$ $\qquad \therefore X = \begin{pmatrix} 0 & \frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

TRY TO SOLVE

Find the matrix X which satisfies that : $X \times \begin{pmatrix} 3 & 7 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 0 & 3 \end{pmatrix}$

Solving two simultaneous equations by using the multiplicative inverse of a matrix

To solve two linear simultaneous equations in the form : $a_1 x + b_1 y = c_1$, $a_2 x + b_2 y = c_2$ by using the multiplicative inverse of a matrix , follow the following :

1 Write the two equations in the form of a matrix equation :

$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ **i.e.** in the form $AX = C$ where

$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is called the matrix of coefficients

, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is called the matrix of variables and $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is called the matrix of constants.

2 Find the solution of the matrix equation :

$AX = C$, then $X = A^{-1} C$ and from that we deduce the values of the variables x and y

Example 5

Solve each system of the following linear equations using the matrices :

1 $2x+3y=7$, $x-y=1$ **2** $x=2y-1$, $3y=2x$

Solution

1 The matrix equation is AX = C , where $A=\begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\end{pmatrix}$, $C=\begin{pmatrix}7\\ 1\end{pmatrix}$

$\because \Delta=|A|=\begin{vmatrix}2 & 3\\ 1 & -1\end{vmatrix}=(2)(-1)-(3)(1)-5\neq 0$

$\therefore$ For the matrix A , there is a multiplicative inverse A^{-1}

$\therefore A^{-1}=\frac{1}{-5}\begin{pmatrix}-1 & -3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}\frac{1}{5} & \frac{3}{5}\\ \frac{1}{5} & \frac{-2}{5}\end{pmatrix}$, $\because X=A^{-1}C$

$\therefore X=\begin{pmatrix}\frac{1}{5} & \frac{3}{5}\\ \frac{1}{5} & \frac{-2}{5}\end{pmatrix}\begin{pmatrix}7\\ 1\end{pmatrix}=\begin{pmatrix}2\\ 1\end{pmatrix}$ $\therefore \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2\\ 1\end{pmatrix}$

$\therefore x=2$, $y=1$, and the solution = $\{(2 , 1)\}$

2 $x-2y=-1$, $2x-3y=0$

The matrix equation is AX = C , where $A=\begin{pmatrix}1 & -2\\ 2 & -3\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\end{pmatrix}$ and $C=\begin{pmatrix}-1\\ 0\end{pmatrix}$

$\because \Delta=|A|=\begin{vmatrix}1 & -2\\ 2 & -3\end{vmatrix}=(1)(-3)-(-2)(2)=1\neq 0$

$\therefore A^{-1}=\frac{1}{1}\begin{pmatrix}-3 & 2\\ -2 & 1\end{pmatrix}=\begin{pmatrix}-3 & 2\\ -2 & 1\end{pmatrix}$

, $\because X=A^{-1}C$ $\therefore X=\begin{pmatrix}-3 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}-1\\ 0\end{pmatrix}=\begin{pmatrix}3\\ 2\end{pmatrix}$

$\therefore \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3\\ 2\end{pmatrix}$ $\therefore x=3$, $y=2$, and the solution set = $\{(3 , 2)\}$

TRY TO SOLVE

Solve the system of the following equations using the matrices :

$x+2y=4$, $y=2x+7$

Example 6

If the curve of the function $f : f(X) = a\,X^2 + b$ passes through the two points $(2 , 0)$ and $(-1 , -3)$ **use the matrices to find the value of the two constants :** a and b

Solution

$\because$ The curve of the function f passes through the point $(2 , 0)$

$\therefore f(2) = 0$ $\qquad\qquad$ $\therefore a \times (2)^2 + b = 0$

$\therefore 4a + b = 0$ $\qquad\qquad$ (1)

, $\because$ the curve of the function f passes through the point $(-1 , -3)$

$\therefore f(-1) = -3$ $\qquad\qquad$ $\therefore a \times (-1)^2 + b = -3$

$\therefore a + b = -3$ $\qquad\qquad$ (2)

To solve the two equations (1) and (2) , we write the matrix equation AX = C ,

where $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} a \\ b \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

$\therefore X = A^{-1} C$

, $\because \Delta = |A| = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} (4)(1) - (1)(1) = 3$

$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{4}{3} \end{pmatrix}$

$\therefore X = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore a = 1 \quad , \quad b = -4$

Remark

We can use the scientific calculator to find the multiplicative inverse of a matrix and we will represent it at the end of the unit.

Technological Activity on Unit One

Using scientific calculator in matrices

We can use scientific calculator which supports matrices for many operations which related to matrices like :

- Finding the transpose of the matrix.
- Performance of adding and subtracting operations on the matrices.
- Finding the value of the determinant.
- Finding the multiplicative inverse of the matrix.

and what we let here , will be by using the calculator of the kind (CASIO fx-991ES PLUS)

First Entering of the matrix $A = \begin{pmatrix} -7 & 0 \\ 4 & 7 \end{pmatrix}$:

- Press successively the following buttons from left to right : → MODE 6 1 5

and this for choosing a matrix of order 2×2 , then enter the elements of the matrix A by pressing successively the following buttons : $\xrightarrow[\text{of the first row}]{\text{Entering of the elements}}$ (-) 7 = 0 =

$\xrightarrow[\text{of the second row}]{\text{Entering of the elements}}$ 4 = 7 =

Second Entering of the matrix $B = \begin{pmatrix} -8 & 4 \\ 0 & 7 \end{pmatrix}$:

- Press successively the following buttons from left to right : → SHIFT [MATRIX] 4 2 2 5

for choosing another matrix of order 2×2 , then enter the elements of the matrix B by pressing successively the following buttons : $\xrightarrow[\text{of the first row}]{\text{Entering of the elements}}$ (-) 8 = 4 =

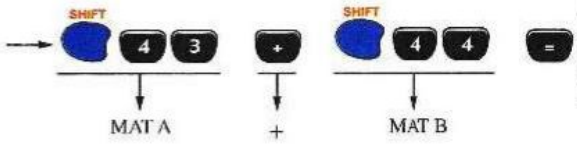
$\xrightarrow[\text{of the second row}]{\text{Entering of the elements}}$ 0 = 7 = AC

Now , we entered the two matrices A and B , and we can do some of the operations on them as the following :

1 To find A^t , press → $\underbrace{\text{SHIFT } 4 \; 8}_{\text{To choose the order Trn (transpose)}}$ $\underbrace{\text{SHIFT } 4 \; 3}_{\text{To choose MAT A (matrix A)}}$ =

successively from left to right :

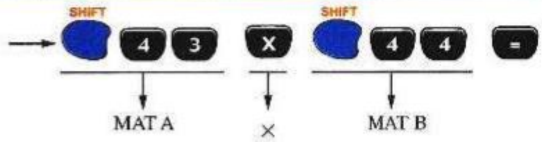
The matrix $\begin{pmatrix} -7 & 4 \\ 0 & 7 \end{pmatrix}$ will appear on the screen which represents A^t

2 To find $A + B$, press 

successively from left to right :

The matrix $\begin{pmatrix} -15 & 4 \\ 4 & 14 \end{pmatrix}$ will appear on the screen which represents $A + B$

---

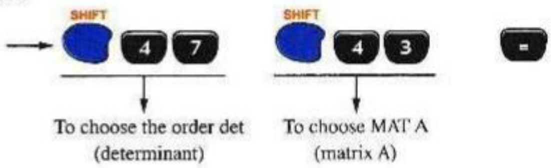
3 To find AB , press 

successively from left to right :

The matrix $\begin{pmatrix} 56 & -28 \\ -32 & 65 \end{pmatrix}$ will appear on the screen which represents AB

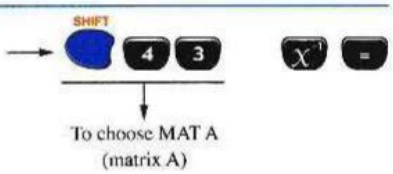
---

4 To find the value of the determinant of the matrix A , press successively from left to right :



-49 will appear on the screen which represents the value of the determinant of the matrix A

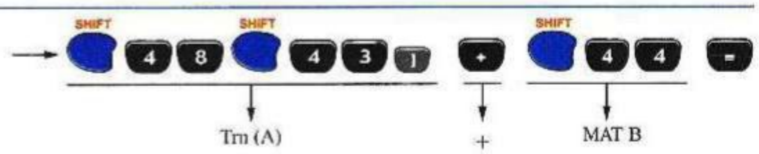
---

5 To find the multiplicative inverse of the matrix A , press 

successively from left to right :

The matrix $\begin{pmatrix} -\frac{1}{7} & 0 \\ \frac{4}{49} & \frac{1}{7} \end{pmatrix}$ will appear on the screen which represents the multiplicative inverse of matrix A

---

6 To find $A^t + B$, press 

successively from left to right :

The matrix $\begin{pmatrix} -15 & 8 \\ 0 & 14 \end{pmatrix}$ will appear on the screen which represents $A^t + B$

TRY TO SOLVE

Use the calculator to find each of the following :

B^t , $A - B$, BA , the determinant B , B^{-1} , $A + B^t$, $A^t B$ and BA^t

Unit 2

Linear programming

Unit Lessons

Lesson 1 — Linear inequalities - Solving systems of linear inequalities graphically.

Lesson 2 — Linear programming and optimization.

Learning outcomes

By the end of this unit, the student should be able to :

- Solve first degree inequalities in one variable and represent the solution graphically.
- Solve first degree inequalities in two variables and determine the region of solution graphically.
- Solve a system of linear inequalities graphically.
- Solve life problems on systems of linear inequalities.
- Use linear programming to solve life mathematical problems.
- Record the data of a mathematical life problem in a suitable table , and transfer these data in the form of linear inequalities , then determine the region of solution graphically.
- Determine the objective function in terms of the coordinates and determine the points which belong to the solution set , giving the optimum solution to the objective function.

Lesson **1**

Linear inequalities - Solving systems of linear inequalities graphically

- Remember the properties of the inequality relation in $\mathbb{R}$:

> Assuming that a , b and c are three real numbers , then :
> - If $a \leq b$, then $a + c \leq b + c$ whether c is positive or negative.
> - If $a \leq b$, then $ac \leq bc$ if c is positive.
> - If $a \leq b$, then $ac \geq bc$ if c is negative.

- You can deduce the previous properties in cases of the other inequality relation signs «$\geq$, $>$, $<$»

Solving the first degree inequality in one variable graphically

- Each of the inequalities :

  $3x < 5$, $4 - x \geq 2x$, $3 \leq x < 6$

  is called an inequality of the first degree in one variable.

- Solving the inequality means finding all the elements of the substitution set which satisfy the inequality.

- The substitution set may be $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$

  and the following illustrative example shows how to solve the first degree inequality in the two cases.

Illustrative example

Show graphically the S.S. of the inequality : $3X + 10 > 1$

1 If the substitution set is $\mathbb{R}$

2 If the substitution set is $\mathbb{R} \times \mathbb{R}$

$3X + 10 > 1$

$\therefore 3X > -9$

$\therefore X > -3$

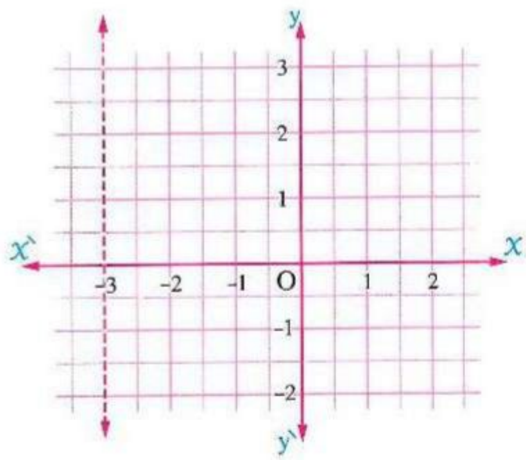
Case (1)

If the substitution set is $\mathbb{R}$, then the S.S. is represented on the number line.

- The S.S. is all the real numbers greater than -3
- The S.S. is the part of the number line on the right of -3
- The unclosed circle at -3 means -3 does not belong to the S.S.

Case (2)

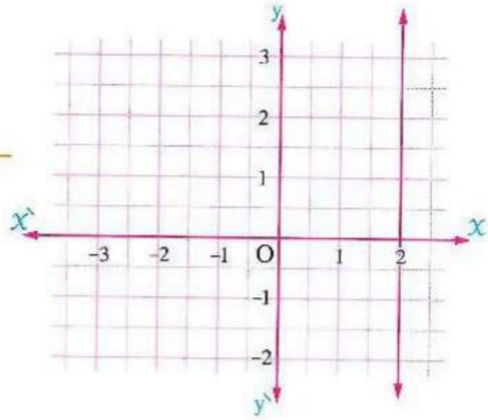
If the substitution set is $\mathbb{R} \times \mathbb{R}$, then the S.S. is represented on a lattice.



- The S.S. is all the ordered pairs whose X-projection is greater than -3
- The S.S. is the region on the right of the straight line $X = -3$ (is called half plane).
- The straight line $X = -3$ is drawn dashed because its points don't belong to the S.S.

Example 1

Show graphically the S.S. of the inequality :
$5x - 7 \leq 2x - 1$ in $\mathbb{R} \times \mathbb{R}$



Solution

$\because 5x - 7 \leq 2x - 1$

$\therefore 5x - 2x \leq -1 + 7$

$\therefore 3x \leq 6$ $\qquad \therefore x \leq 2$

Notice that

1. The shaded region is on the left of the straight line $x = 2$ because the inequality relation is "smaller than".
2. The straight line $x = 2$ is drawn solid because the inequality contains the symbol of equality **i.e.** $\leq$

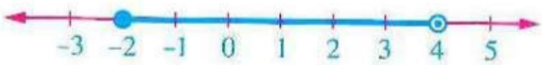
Example 2

Show graphically the S.S. of the inequality : $x - 1 \leq 4x + 5 < x + 17$ where $x \in \mathbb{R}$

Solution

$\because x - 1 \leq 4x + 5 < x + 17$ $\qquad \therefore -1 \leq 3x + 5 < 17$ $\qquad \therefore -6 \leq 3x < 12$

$\therefore -2 \leq x < 4$ $\qquad \therefore$ The S.S. $= [-2 , 4[$



Example 3

Find graphically the S.S. of the inequality : $2x - 2 \leq 3x - 1 < x + 5$ where $x \in \mathbb{R}$

Solution

Parting the inequality into two inequalities as the following :

$2x - 2 \leq 3x - 1$

$\therefore 2x - 3x \leq -1 + 2$

$\therefore -x \leq 1$ $\qquad \therefore x \geq -1$

$\therefore$ The S.S. $= [-1 , \infty[$

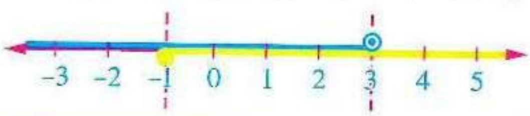
$3x - 1 < x + 5$

$\therefore 3x - x < 5 + 1$

$\therefore 2x < 6$ $\qquad \therefore x < 3$

$\therefore$ The S.S. $=]-\infty , 3[$

$\therefore$ The S.S. of the original inequality $= [-1 , \infty[\cap]-\infty , 3[= [-1 , 3[$

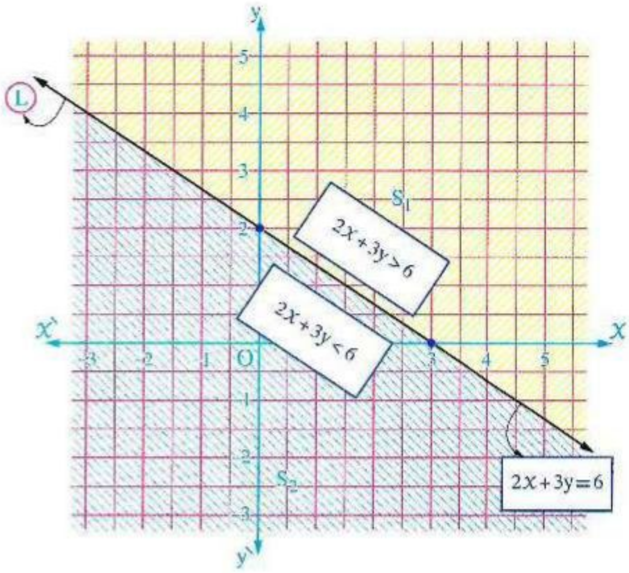


Solving the first degree inequality in two variables graphically

- We know that we can represent the linear equation : $2x + 3y = 6$ graphically by a straight line as follows :

x	0	3
y	2	0

«We should get a third ordered pair to check the graph»



- From the graph , we notice that this straight line divides the Cartesian plane into three sets of points :

1. The set of points of the straight line L (is called a boundary line) and each of these points satisfies that $2x + 3y = 6$
2. The set of points of the plane that lies on one side of the straight line L (and it is called a half plane) and is denoted by S_1 and each of them satisfies that $2x + 3y > 6$
3. The set of points of the plane that lies on the other side of the straight line L (and it is called a half plane also) and is denoted by S_2 and each of them satisfies that $2x + 3y < 6$

From the previous , we deduce that

- The half plane S_1 is the region representing the S.S. of the inequality : $2x + 3y > 6$
- The union of the points of the half plane S_1 and the straight line L represents the S.S. of the inequality : $2x + 3y \geq 6$
- The half plane S_2 is the region representing the S.S. of the inequality : $2x + 3y < 6$
- The union of the points of the half plane S_2 and the straight line L represents the S.S. of the inequality : $2x + 3y \leq 6$

Steps of solving the first degree inequality in two variables graphically

1. Represent the straight line equation related to the inequality by a solid line in case of ≥ or ≤ , and by a dashed line in case of > or <
2. Determine the half plane in which the feasible "or solution" region lies by choosing any point (x_1 , y_1) belonging to one half plane as a test point and substitute it in the inequality.
   - If the chosen point satisfied the inequality, then the half plane containing this point is the feasible region of the inequality.
   - If the chosen point did not satisfy the inequality, then the other half plane is the feasible region of the inequality.

Remark

To make it easier, choose the origin point (0 , 0) if the boundary line does not pass through it.

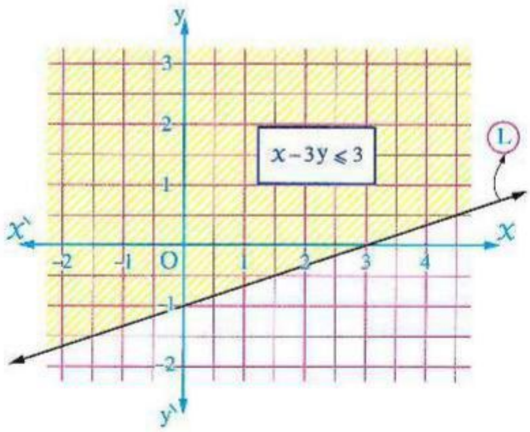
Example 4

Represent graphically the S.S. of the inequality : $x - 3y \leq 3$ in $\mathbb{R} \times \mathbb{R}$

Solution

1. Draw the boundary line L whose equation is :
   $x - 3y = 3$ as a solid straight line
   because the inequality relation is ≤
   using the following table :

x	0	3
y	-1	0



2. Choose the origin point as a test point.
   ∵ The point (0 , 0) satisfies the inequality (because 0 < 3)
   ∴ The S.S. of the inequality is the boundary line L ∪ the half plane that contains the point (0 , 0) and that is represented by the shaded region in the previous graph.

Notice that

You can draw the boundary line L without the previous table by using the slope of the straight line and the intercepted part of the y-axis as you studied before.

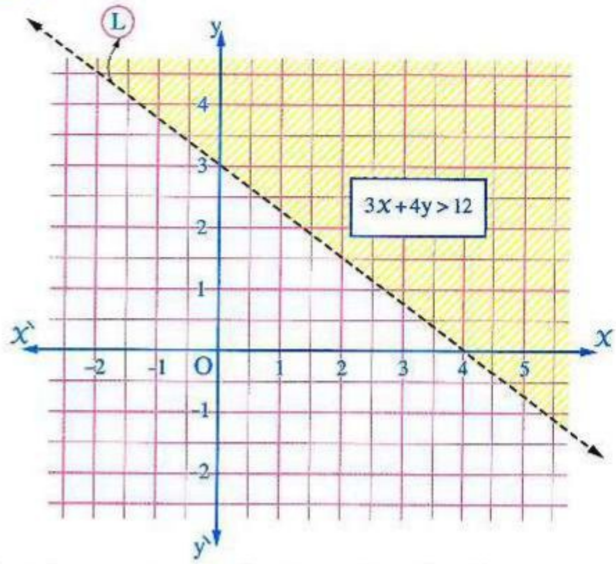
Example 5

Represent graphically the S.S. of the inequality : $3x + 4y > 12$ in $\mathbb{R} \times \mathbb{R}$

Solution

1 Draw the boundary line L whose equation is : $3x + 4y = 12$ as a dashed line because the inequality relation is > using the following table :

x	0	4
y	3	0



2 Choose the origin point as a test point.

, $\because$ (0 , 0) does not satisfy the inequality (because $0 < 12$)

$\therefore$ The S.S. of the inequality is the half plane that does not contain the point (0 , 0) and is represented by the shaded region in the graph.

Remarks

- The equation : $y = 0$ is represented by the x-axis.
- The equation : $x = 0$ is represented by the y-axis.
- The equation : $y = a$ is represented by a straight line parallel to the x-axis and passing through the point (0 , a)
- The equation : $x = a$ is represented by a straight line parallel to the y-axis and passing through the point (a , 0)
- The straight line whose equation is in the form : $\frac{x}{a} + \frac{y}{b} = 1$ passes through the two points (a , 0) and (0 , b)

TRY TO SOLVE

Represent graphically the S.S. of the inequality : $2x - 5y \leq 10$ in $\mathbb{R} \times \mathbb{R}$

Solving systems of linear inequalities graphically

To find the graphical solution of two inequalities , we do as the following :

1 We shade the region S_1 that represents the S.S. of the 1^{st} inequality.

2 We shade the region S_2 that represents the S.S. of the 2^{nd} inequality.

- The common region S of the two shaded regions S_1 and S_2 represents the S.S. of the two inequalities where $S = S_1 \cap S_2$

Example 6

Represent graphically the S.S. of the two inequalities :
$x + 3y \le 3$, $2x + y \le 4$ in $\mathbb{R} \times \mathbb{R}$

Solution

1 Draw the boundary line $L_1 : x + 3y = 3$ as a solid line using the following table :

x	0	3
y	1	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 3$)

$\therefore$ The region S_1 is the S.S. of the inequality : $x + 3y \le 3$

and it is represented by $L_1 \cup$ the half plane in which the origin point lies [***Fig. (1)***]

2 Draw the boundary line $L_2 : 2x + y = 4$ as a solid line using the following table :

x	0	2
y	4	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 4$)

$\therefore$ The region S_2 is the S.S. of the inequality : $2x + y \le 4$

and it is represented by $L_2 \cup$ the half plane in which the origin point lies [***Fig. (2)***]

3 The S.S. of the two inequalities simultaneously is $S = S_1 \cap S_2$ and it is represented by the common region in the two shaded parts [***Fig. (3)***]

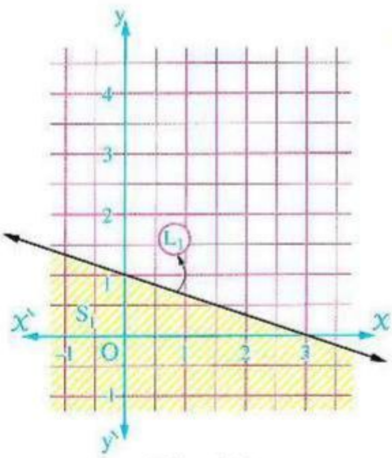


Fig. (1)

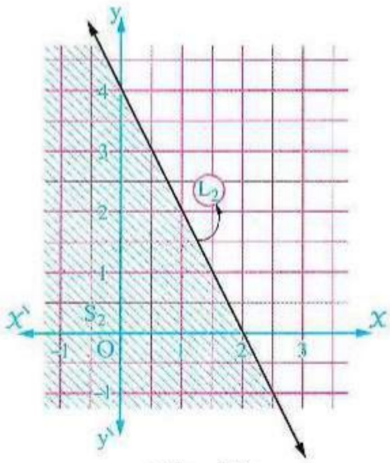


Fig. (2)

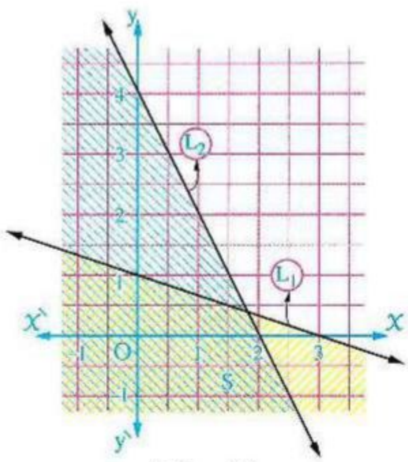


Fig. (3)

Remark

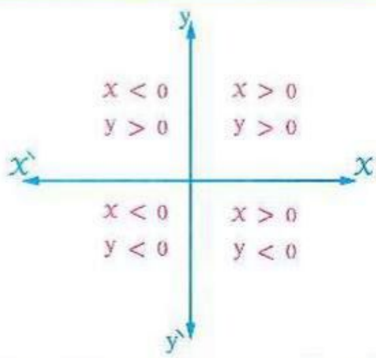
The two coordinate axes divide the Cartesian plane into four quadrants :

1^{st} quadrant : where $x > 0$, $y > 0$

, 2^{nd} quadrant : where $x < 0$, $y > 0$

, 3^{rd} quadrant : where $x < 0$, $y < 0$

and 4^{th} quadrant : where $x > 0$, $y < 0$



Example 7

Represent graphically the S.S. of the inequalities :

$x \geq 0$, $y \geq 0$, $y + 3x \leq 9$ and $y - x < 1$ in $\mathbb{R} \times \mathbb{R}$

Solution

1 The S.S. of the two inequalities : $x \geq 0$ and $y \geq 0$

is represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the 1st quadrant of the Cartesian plane.

2 Draw the boundary line $L_1 : y + 3x = 9$ as a solid line.

x	2	3
y	3	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 9$)

$\therefore$ The region S_1 is the S.S. of the inequality : $y + 3x \leq 9$ and it is represented by $L_1 \cup$ the half plane in which the point (0 , 0) lies [***Fig. (1)***]

3 Draw the boundary line $L_2 : y - x = 1$ as a dashed line.

x	0	-1
y	1	0

$\because$ The point (0 , 0) satisfies the inequality (because $0 < 1$)

$\therefore$ The region S_2 is the S.S. of the inequality : $y - x < 1$ and it is represented by the half plane in which the point (0 , 0) lies [***Fig. (2)***]

4 S is the S.S. of the four inequalities which is represented by the region in the 1st quadrant that has the common shade [***Fig. (3)***]

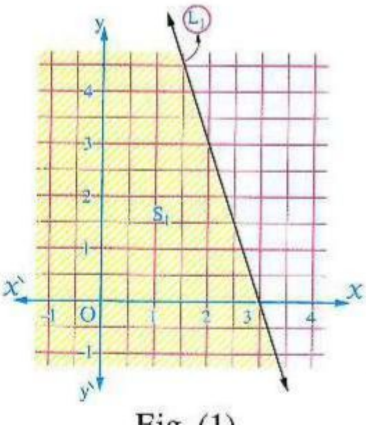


Fig. (1)

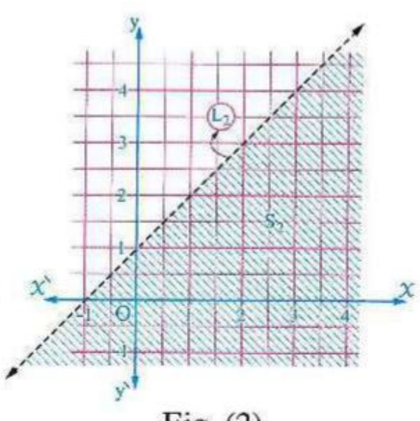


Fig. (2)

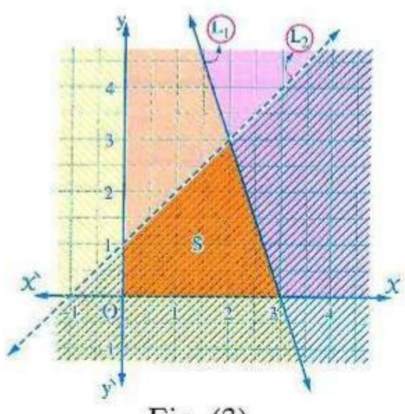


Fig. (3)

Remark

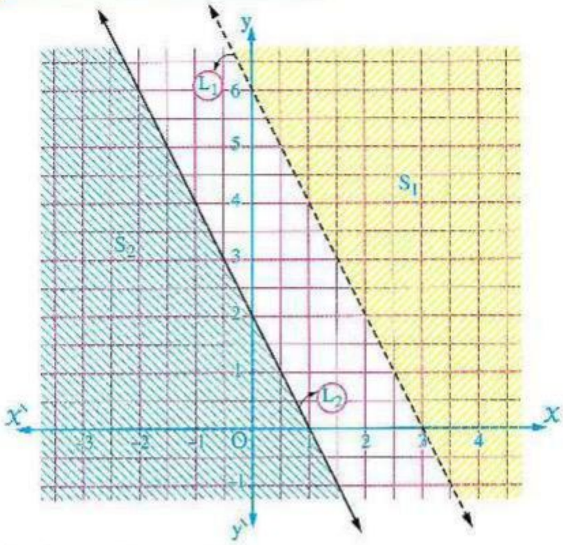
In the previous two examples , we draw a separate figure to show the feasible region of each inequality. After that we deduced the last figure which shows the feasible region of all the inequalities simultaneously. You (after some practice) won't be in need of drawing all these figures, but you will satisfy the last figure only.

Example 8

Represent graphically the S.S. of the inequalities :

$2x + y > 6$, $4x + 2y \leq 4$ in $\mathbb{R} \times \mathbb{R}$

Solution



1 Draw the boundary line $L_1 : 2x + y = 6$ as a dashed line that passes through the two points (0 , 6) and (3 , 0)

, ∵ the point (0 , 0) does not satisfy the inequality.

∴ The region S_1 is the S.S. of the inequality : $2x + y > 6$ and it is represented by the half plane which does not contain the origin point.

2 Draw the boundary line $L_2 : 4x + 2y = 4$ as a solid line that passes through the two points (0 , 2) and (1 , 0)

, ∵ the point (0 , 0) satisfies the inequality.

∴ The region S_2 is the S.S of the inequality : $4x + 2y \leq 4$ and it is represented by $L_2 \cup$ the half plane which contains the origin point.

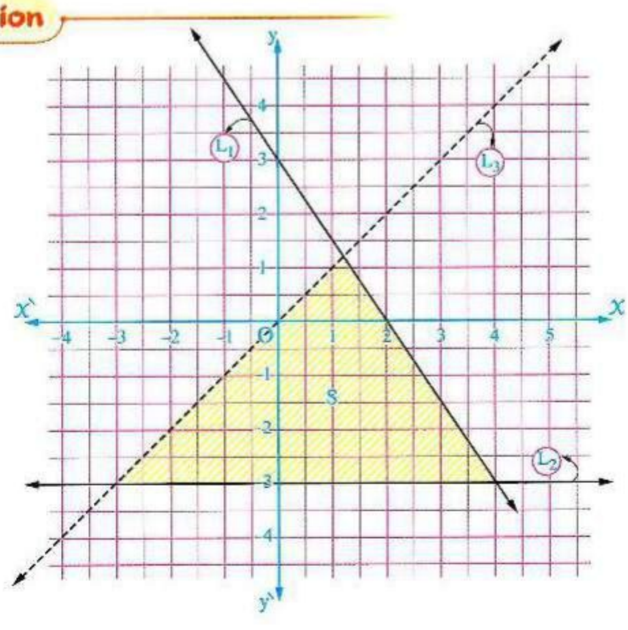
3 The S.S. of the two inequalities simultaneously is $S = S_1 \cap S_2 = \varnothing$

Example 9

Represent graphically the S.S. of the following inequalities :

$3x + 2y \leq 6$, $y + 3 \geq 0$ and $x - y > 0$ in $\mathbb{R} \times \mathbb{R}$

Solution



1 Draw the boundary line $L_1 : 3x + 2y = 6$ as a solid line that passes through the two points (2 , 0) and (0 , 3)

, ∵ the point (0 , 0) satisfies the inequality (because $0 < 6$)

∴ The S.S. (S_1) is represented by $L_1 \cup$ the half plane in which the origin point lies.

2 Draw the boundary line $L_2 : y = -3$ as a solid line [A straight line is parallel to the X-axis and passes through the point $(0 , -3)$]

, $\because$ the point $(0 , 0)$ satisfies the inequality (because $0 > -3$)

$\therefore$ The S.S. (S_2) is represented by $L_2 \cup$ the half plane in which the origin point lies.

3 Draw the boundary line $L_3 : X - y = 0$

as a dashed line that passes through the two points $(0 , 0)$ and $(1 , 1)$

, $\because$ the point $(0 , 2)$ does not satisfy the inequality (because $-2 \not> 0$)

$\therefore$ The S.S. (S_3) is represented by the half plane in which the point $(0 , 2)$ does not lie.

4 The S.S. of the three inequalities simultaneously is $S = S_1 \cap S_2 \cap S_3$

and it is represented by the shaded region in the shown Cartesian plane.

Example 10

A factory for children toys produces cars and planes. It produces 250 toys daily at most. If the cost of one car is L.E. 15 and of one plane is L.E. 10 and the total cost of the daily production is not more than L.E. 3000 , write a system of linear inequalities representing the previous , then represent graphically the solution region of this system.

Solution

Let the number of cars = X , the number of planes = y

- The system of inequalities is :

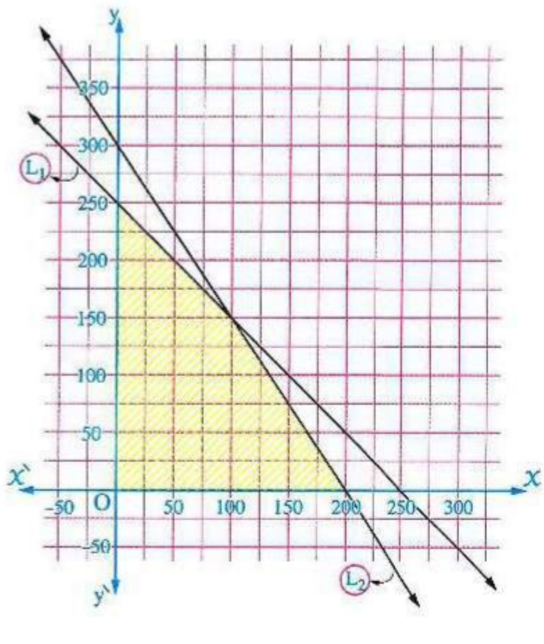
1 $X \geq 0$ **2** $y \geq 0$

3 $X + y \leq 250$

4 $15 X + 10 y \leq 3000$ **i.e.** $3 X + 2 y \leq 600$

- Determining the region which represents the S.S. of the inequalities as follows :

1 The inequalities $X \geq 0$, $y \geq 0$ are represented by $\overrightarrow{OX} \cup \overrightarrow{Oy} \cup$ the 1^{st} quadrant.



2 Draw the boundary line $L_1 : x + y = 250$ as a solid line that passes through the two points $(0 , 250)$ and $(250 , 0)$

, $\because$ the point $(0 , 0)$ satisfies the inequality (because $0 < 250$)

$\therefore$ The S.S. of this inequality is represented by $L_1 \cup$ the half plane in which the origin point lies.

3 Draw the boundary line $L_2 : 3x + 2y = 600$ as a solid line that passes through the two points $(0 , 300)$ and $(200 , 0)$

, $\because$ the point $(0 , 0)$ satisfies the inequality (because $0 < 600$)

$\therefore$ The S.S. of this inequality is represented by $L_2 \cup$ the half plane in which the origin point lies.

4 The ordered pairs that its x-coordinates and y-coordinates are integers in the shaded region is the S.S. of the required system of linear inequalities.

Lesson 2

Linear programming and optimization

Linear programming

It is one of the scientific methods that is used to give the best decision of solving a problem or it is the optimal solution that satisfies a certain object in view of some restrictions and available abilities or materials where the object can be put in the form of a linear function called "the objective function" and the stipulations and available abilities are put in the form of linear inequalities.

The method of linear programming depends on :

1 Representing the system of inequalities that expresses the stipulations such that we obtain a ribbed region representing the S.S. of the inequalities.

Often the restrictions contain the two inequalities : $x \geq 0$, $y \geq 0$, that means the S.S. (the region representing the S.S.) lies in the first quadrant.

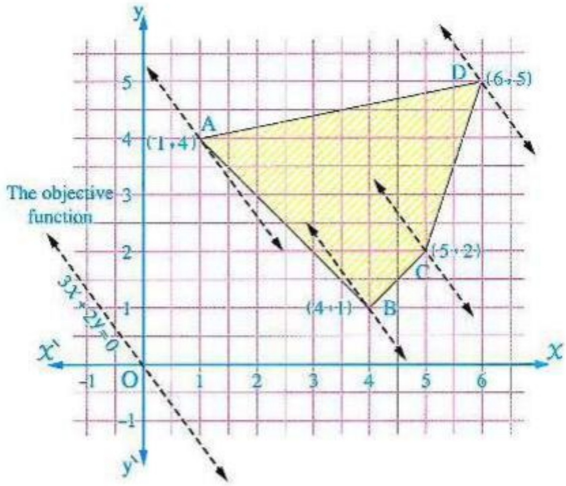
2 Determining the objective function in the form $P = \ell x + m y$ where ℓ and m are constants we represent the equation $\ell x + m y = 0$ by a straight line that passes through the origin point , then we let this straight line move parallel to itself upwards till it passes through the vertices of the polygon that determines the region of the S.S.

Since all these parallel straight lines have the same slope and differ only in the value of "P" and each point (x , y) belonging to the S.S. and to the same straight line gives a value to the number "P"

So , we can determine the greatest value or the smallest value of the objective function.

For example :

If the S.S. representing the set of inequalities that represents the restrictions is the shaded region in the opposite graph and the required is finding the greatest and smallest value of the expression $P = 3x + 2y$, then we substitute by the coordinates of the points : A , B , C and D "the vertices of the polygon" in the objective function.



Notice that

The value of the objective function at any point that lies on a side of the shaded region sides is included between its values at the two vertices of the polygon for the side that joins them.

$\therefore [P]_A = 3 \times 1 + 2 \times 4 = 11$, $[P]_B = 3 \times 4 + 2 \times 1 = 14$

, $[P]_C = 3 \times 5 + 2 \times 2 = 19$, $[P]_D = 3 \times 6 + 2 \times 5 = 28$

So , we find that the greatest value is 28 at the vertex D (6 , 5) and the smallest value is 11 at the vertex A (1 , 4)

Example 1

Determine the S.S. of the following inequalities simultaneously :

$x \geq 0$, $y \geq 0$, $x + 2y \leq 8$ and $3x + 2y \leq 12$

Then find from the S.S. (x , y) that makes "P" maximum where $P = 50x + 75y$

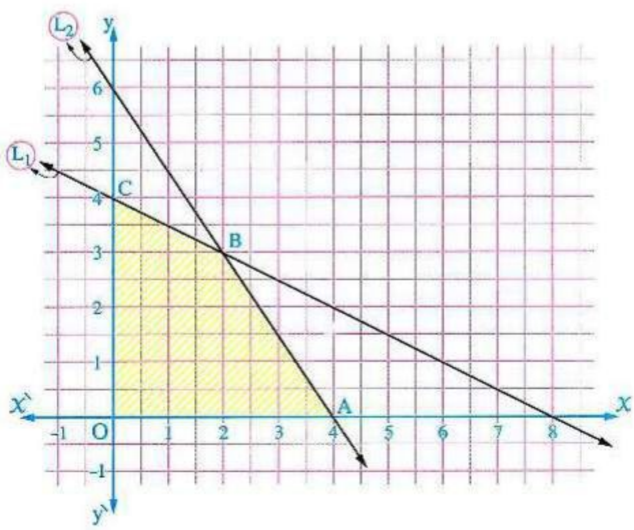
Solution

First Determine the region that represents the S.S. of the inequalities :

1 The two inequalities : $x \geq 0$ and $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

2 Draw the boundary line $L_1 : x + 2y = 8$ (as a solid line) that passes through the two points (0 , 4) and (8 , 0)

3 Draw the boundary line $L_2 : 3x + 2y = 12$ (as a solid line) that passes through the two points (0 , 6) and (4 , 0)



$\therefore$ The solution set of the inequalities is represented by the shaded region. That is the ribbed region ABCO

To get the coordinates of the point B algebraically : Solve the two equations representing the two straight lines L_1 and L_2 simultaneously where :
$L_1 : x + 2y = 8$, $L_2 : 3x + 2y = 12$
, then we find that : B = (2 , 3)

Second Determine the vertices of the feasible region :

The vertices of the feasible region are : A (4 , 0) , B (2 , 3) , C (0 , 4) and O (0 , 0)

Third Determine the value of the objective function at each vertex :

$\because$ The objective function is : $P = 50x + 75y$

$\therefore [P]_A = 50 \times 4 + 75 \times 0 = 200$, $[P]_B = 50 \times 2 + 75 \times 3 = 325$

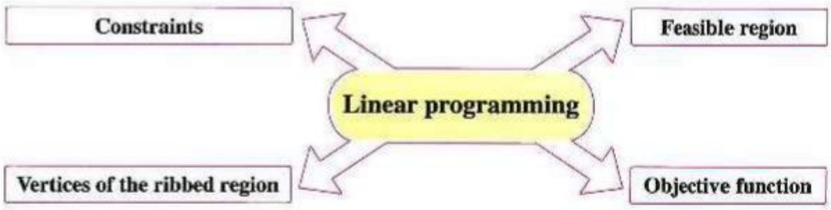
, $[P]_C = 50 \times 0 + 75 \times 4 = 300$, $[P]_O = 50 \times 0 + 75 \times 0 = 0$

$\therefore$ The maximum value of the function P is 325 at the point B (2 , 3)

Life applications on linear programming

We can deal with the life problems which are related to the linear programming by the following steps :

1. Analyse the situation or the problem to determine the variables , the constraints and the available data and arrange them in a table.
2. Put the constraints in the form of a system of linear inequalities.
3. Write the objective function.
4. Represent the system of linear inequalities graphically and determine the feasible region.
5. Determine the vertices of the feasible region.
6. Find the objective function at each vertex of the previous vertices to determine the vertex where the required objective satisfied at it.



Example 2

A bakery produces two kinds of cake. The first kind of cake needs 200 gm. of flour and 25 gm. of butter and the second kind of cake needs 100 gm. of flour and 50 gm. of butter. If the quantity of the given flour is 4 kg. and the given butter is $1\frac{1}{4}$ kg. ,
find the greatest possible number of cakes that can be made.

Solution

- Let the number of cakes of the first kind be x

and the number of cakes of the second kind be y

- **Arrange the available data of the problem :**

	1^{st} kind	2^{nd} kind	Given quantity
Flour	200	100	4 000
Butter	25	50	1 250

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $x \geq 0$, $y \geq 0$

2 $200\,x + 100\,y \leq 4\,000$ i.e. $2\,x + y \leq 40$

3 $25\,x + 50\,y \leq 1\,250$ i.e. $x + 2\,y \leq 50$

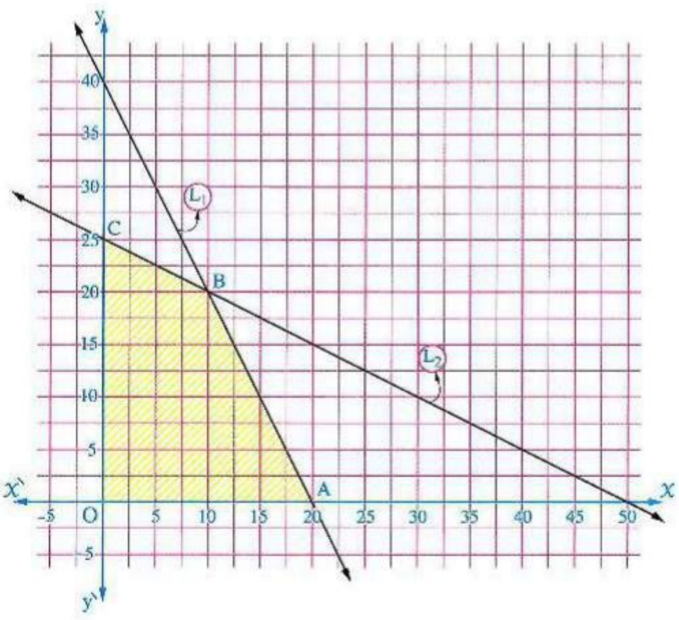
- Write the objective function : $P = x + y$, where P is maximum.

- **Representing the system of linear inequalities graphically and determining the feasible region :**

1 The two inequalities : $x \geq 0$ and $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

2 Draw the boundary line $L_1 : 2\,x + y = 40$ (as a solid line) that passes through the two points (0 , 40) and (20 , 0)

3 Draw the boundary line $L_2 : x + 2\,y = 50$ (as a solid line) that passes through the two points (0 , 25) and (50 , 0)



$\therefore$ The solution set of the inequalities is represented by the shaded region in the opposite graph and this is the ribbed region ABCO

- **Determine the vertices of the feasible region :**

The vertices of the feasible region are : A (20 , 0) , B (10 , 20) , C (0 , 25) and O (0 , 0)

- **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = x + y$

$[P]_O = 0 + 0 = 0$, $[P]_A = 20 + 0 = 20$

, $[P]_B = 10 + 20 = 30$, $[P]_C = 0 + 25 = 25$

$\therefore$ The greatest number of cakes is 30 ones , 10 of the first kind and 20 of the second kind.

Example 3

A factory produces 120 units at most of two different kinds of goods and achieves a profit in each unit of the first kind L.E. 15 and of the second kind L.E. 8 in each unit and the sold quantity of the second kind is not less than half the sold quantity of the first kind.

Find the number of produced units of each kind to satisfy the maximum profit.

Solution

- Let the number of produced units of the first kind be x and the number of produced units of the second kind be y

- **Arrange the available data of the problem in the table :**

	1st kind	2nd kind	The upper limit
The produced units	x	y	120
The profit	15	8	–

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $x \geq 0$, $y \geq 0$

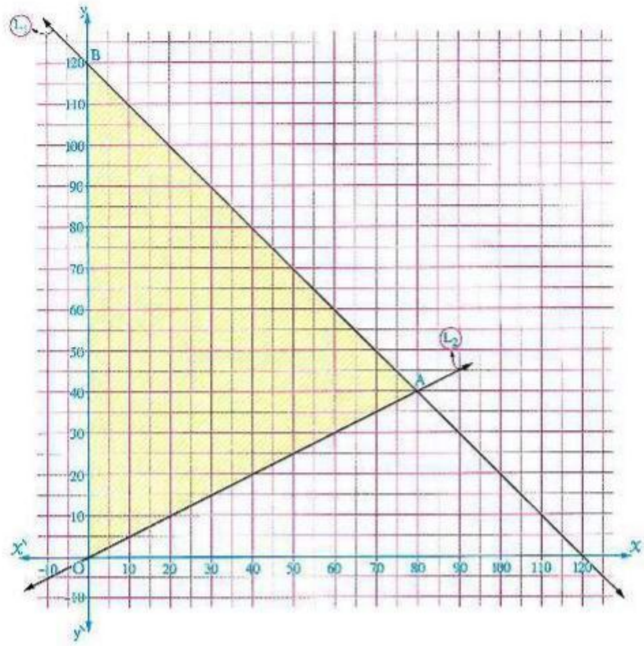
2 $x + y \leq 120$

3 $\because$ y is not less than $\frac{1}{2}x$

$\therefore y \geq \frac{1}{2}x$ $\therefore y - \frac{1}{2}x \geq 0$ $\therefore 2y - x \geq 0$

- Write the objective function : $P = 15x + 8y$, where P is maximum.

• Representing the system of linear inequalities graphically and determining the feasible region :



1 The two inequalities :

$x \geq 0$, $y \geq 0$ are represented by $\overrightarrow{Ox}$ $\cup$ $\overrightarrow{Oy}$ $\cup$ the first quadrant.

2 Draw the boundary line

$L_1 : x + y = 120$ (as a solid line) that passes through the two points (0 , 120) and (120 , 0)

3 Draw the boundary line

$L_2 : 2y - x = 0$ (as a solid line) that passes through the two points (0 , 0) and (20 , 10)

$\therefore$ The solution set of the inequalities is represented by the shaded region in the opposite graph and this is the triangular region OAB

• Determine the vertices of the feasible region :

The vertices of the feasible region are : O (0 , 0) , A (80 , 40) and B (0 , 120)

• Determine the value of the objective function at each vertex :

$\because$ The objective function is : $P = 15x + 8y$

$[P]_O = 0 + 0 = 0$, $[P]_A = 15 \times 80 + 8 \times 40 = 1\,520$

, $[P]_B = 15 \times 0 + 8 \times 120 = 960$

$\therefore$ The maximum profit that can be achieved is L.E. 1 520 that happens when the production is 80 units of the first kind and 40 units of the second kind.

Example 4

The required is forming a meal consisting of two kinds of food , if the piece of the first kind contains 3 calories , 6 units of vitamin "C" and the piece of the second kind contains 6 calories , 4 units of vitamin "C" Given that we need at least 36 calories and 48 units of vitamin "C" in the meal. If the price of the piece of the first kind is 3 pounds and of the second kind is 4 pounds , **then what is the number of pieces of the meal that satisfies the least limit with the least cost ?**

Solution

- Let the number of pieces of the first kind in the meal be x and the number of pieces of the second kind in the meal be y

- **Arrange the data in a table :**

	Pieces of the first kind	Pieces of the second kind	Least limit
Calories	3	6	36
Vitamin "C"	6	4	48

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $x \geq 0$, $y \geq 0$

2 $3x + 6y \geq 36$ **i.e.** $x + 2y \geq 12$

3 $6x + 4y \geq 48$ **i.e.** $3x + 2y \geq 24$

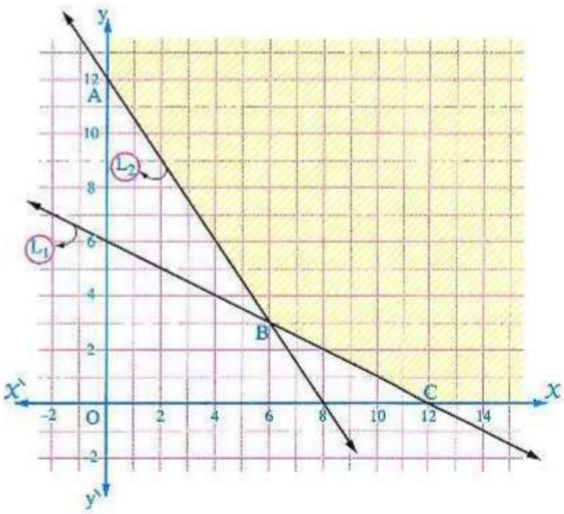
- Write the objective function : $P = 3x + 4y$, where P is minimum.

- **Representing the system of linear inequalities graphically and determining the feasible region :**

1 The two inequalities : $x \geq 0$ and $y \geq 0$ are represented by $\overrightarrow{Ox} \cup \overrightarrow{Oy} \cup$ the first quadrant.

2 Draw the boundary line $L_1 : x + 2y = 12$ (as a solid line) that passes through the two points (0 , 6) and (12 , 0)

3 Draw the boundary line $L_2 : 3x + 2y = 24$ (as a solid line) that passes through the two points (0 , 12) and (8 , 0)



- **Determine the vertices of the feasible region :**

The vertices of the feasible region are : A (0 , 12) , B (6 , 3) and C (12 , 0)

- **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = 3x + 4y$

$\therefore [P]_A = 3 \times 0 + 4 \times 12 = 48$, $[P]_B = 3 \times 6 + 4 \times 3 = 30$

, $[P]_C = 3 \times 12 + 4 \times 0 = 36$

$\therefore$ The least cost of the meal is 30 pounds when it consists of 6 pieces of the first kind and 3 pieces of the second kind.

Example 5

A tourism company aims to rent a fleet of airplanes to transport 2800 passengers , 128 tons of luggage at least and the available kinds of airplanes are A and B and the number of available airplanes of kind (A) is 13 and of kind (B) is 12 and the completed load of the airplane of kind (A) is 200 passengers , 8 tons of luggage and of kind (B) is 100 passengers , 6 tons of luggage , if the rent of airplane of kind (A) is 240 thousand pounds and of kind (B) is 100 thousand pounds , **then how many airplanes of each kind can be rented to satisfy the aim with the least cost ?**

Solution

- Let the number of airplanes of kind A be X and the number of airplanes of kind B be y
- **Arrange the available data of the problem in a table :**

	Kind (A)	Kind (B)	Least limit
Number of passengers	200	100	2800
Luggage in tons	8	6	128

- **Translate the data and the constraints in the form of a system of inequalities :**

1 $X \leq 13$, $y \leq 12$

2 $200\,X + 100\,y \geq 2800$ i.e. $2\,X + y \geq 28$

3 $8\,X + 6\,y \geq 128$ i.e. $4\,X + 3\,y \geq 64$

- Write the objective function :

$P = 240\,X + 100\,y$ where P is minimum.

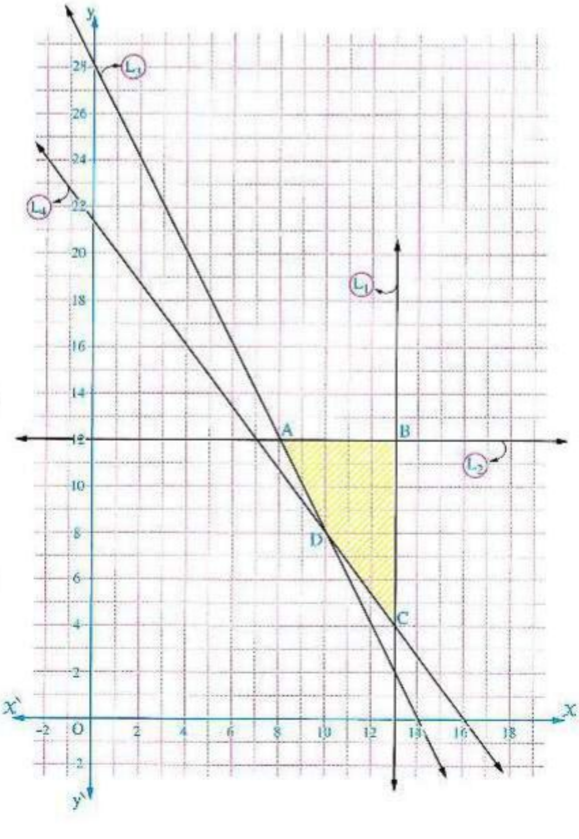
- **Representing the system of linear inequalities graphically and determining the feasible region :**

1 Draw the boundary line

$L_1 : X = 13$ (as a solid line) that is parallel to the y-axis and cuts the X-axis at the point (13 , 0)

2 Draw the boundary line

$L_2 : y = 12$ (as a solid line) that is parallel to the X-axis and cuts the y-axis at the point (0 , 12)



3 Draw the boundary line

$L_3 : 2\,X + y = 28$ (as a solid line) that passes through the two points (0 , 28) and (14 , 0)

4 Draw the boundary line

$L_4 : 4\,X + 3\,y = 64$ (as a solid line) that passes through the two points (1 , 20) and (16 , 0)

• **Determine the vertices of the feasible region :**

The vertices of the feasible region are : A (8 , 12) , B (13 , 12) , C (13 , 4) and D (10 , 8)

• **Determine the value of the objective function at each vertex :**

$\because$ The objective function is : $P = 240\,X + 100\,y$

$\therefore [P]_A = 240 \times 8 + 100 \times 12 = 3120$, $[P]_B = 240 \times 13 + 100 \times 12 = 4320$

, $[P]_C = 240 \times 13 + 100 \times 4 = 3520$, $[P]_D = 240 \times 10 + 100 \times 8 = 3200$

$\therefore$ The least cost that satisfies the aim is the rent of 8 airplanes of kind (A) ,

12 airplanes of kind (B) , and the cost is 3120 thousand pounds.

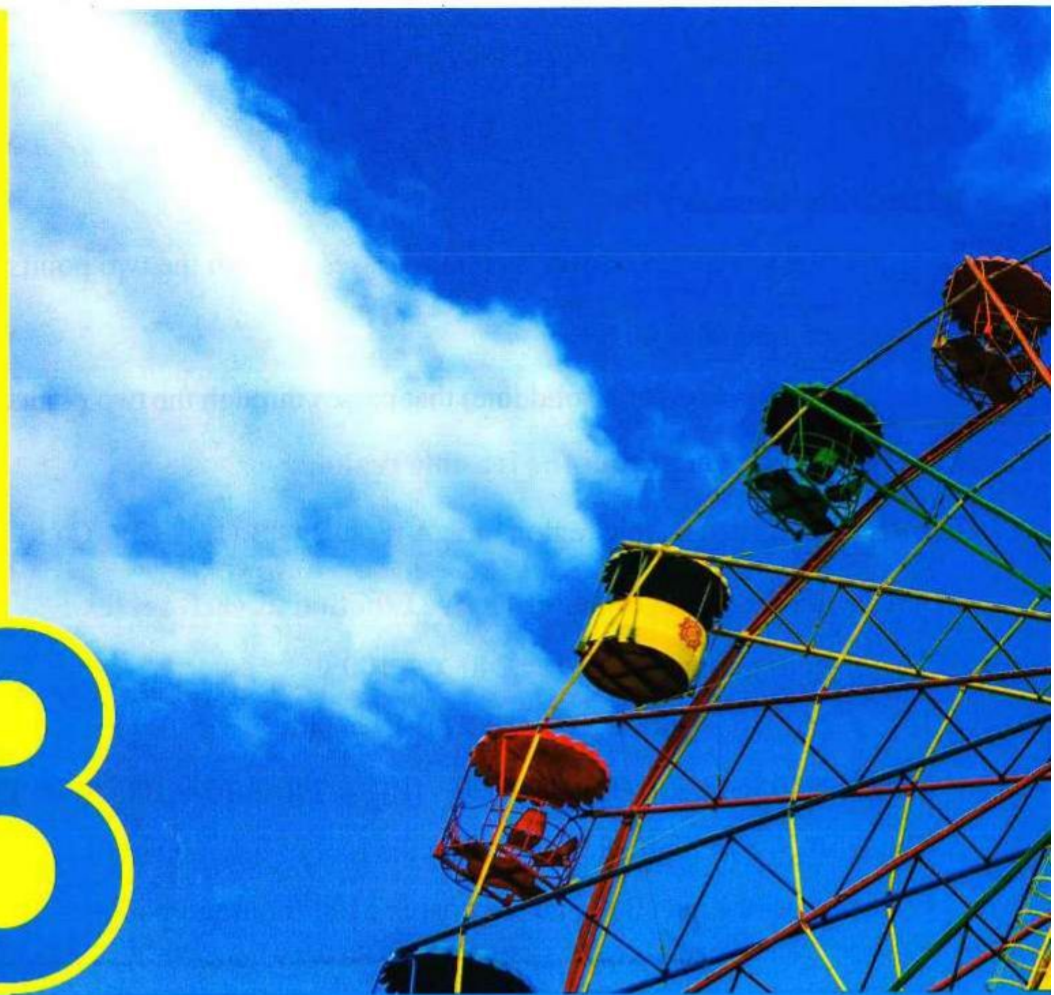
TRY TO SOLVE

A factory produces two kinds of accessories A and B

To produce a piece of the kind A , the factory needs to run two machines , the first for one hour and the second for 2 hours and half. To produce a piece of the kind B , the factory needs running of the first machine for 4 hours and the second for 2 hours. If the first machine does not work more than 8 hours and the second does not work more than 21 hours daily and the profit of the factory is L.E. 24 and L.E. 40 in each piece of the two kinds A and B respectively.

Find the maximum profit the factory can achieve in one day.

Unit 3



Trigonometry

Unit Lessons

- Lesson 1 — Trigonometric identities.
- Lesson 2 — Solving trigonometric equations.
- Lesson 3 — Solving the right-angled triangle.
- Lesson 4 — Angles of elevation and angles of depression.
- Lesson 5 — Circular sector.
- Lesson 6 — Circular segment.
- Lesson 7 — Areas.

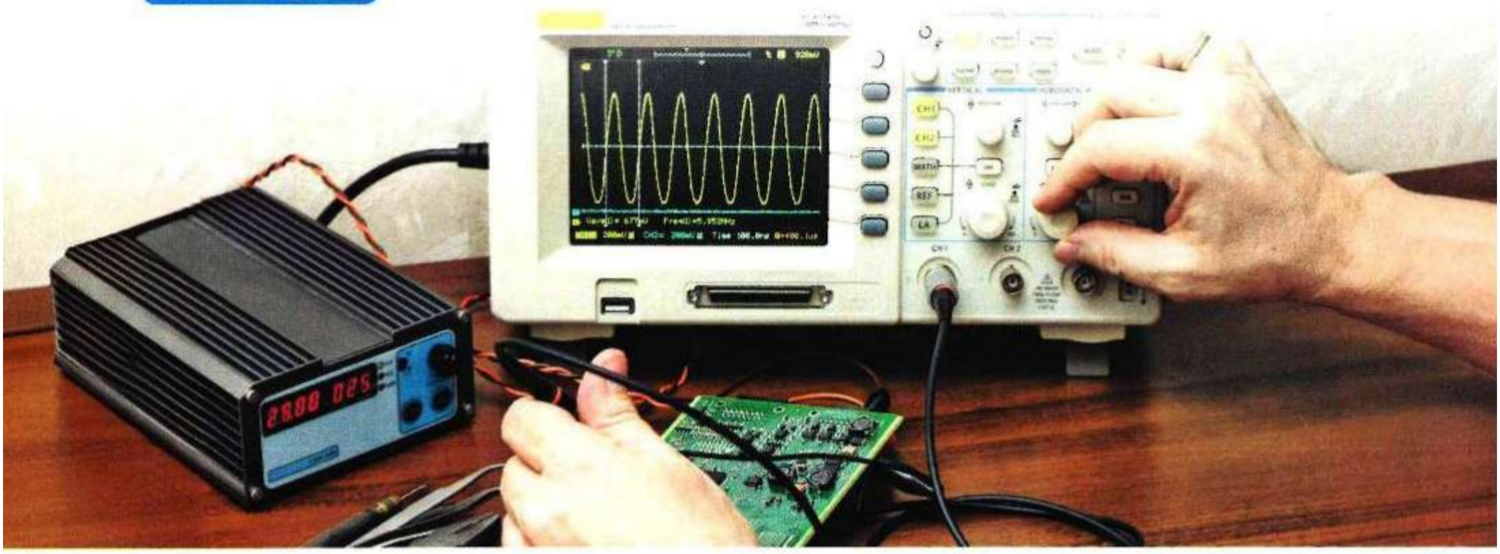
Learning outcomes

By the end of this unit, the student should be able to :

- Deduce the basic relations among trigonometric functions.
- Prove the validity of identities on trigonometric functions.
- Determine the equality if it is identity or trigonometric equation.
- Solve simple trigonometric equations in the general form in the interval $[0 , 2\pi[$
- Recognize the general solution for the trigonometric equation.
- Solve the right-angled triangle.
- Solve applications that involve angles of elevation and depression.
- Recognize the circular sector and how to find its area.
- Recognize the circular segment and how to find its area.
- Find the area of the triangle , the area of the quadrilateral and the area of the regular polygon.
- Use activities for computer programs.

Lesson **1**

Trigonometric identities



Trigonometric identities and equations

The identity

- It is a true equality for all real values of the variable , in which each of the two sides of the equality is known.

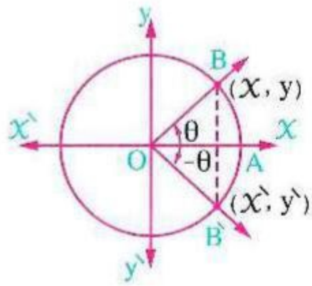
For example :

The equality : $\cos(-\theta) = \cos\theta$ is called identity because it is true for all real values of the variable θ because ,

In the opposite figure :

From our previous study for the related angles θ and $(-\theta)$, we find that :

The point $\grave{B}(\grave{x}, \grave{y})$ is the image of the point $B(x, y)$ by the reflection in the x-axis $\overleftrightarrow{\grave{x}x}$



i.e. $\grave{x} = x$, $\because \cos(-\theta) = \grave{x}$, $\cos\theta = x$

$\therefore \cos(-\theta) = \cos\theta$ for all real values of θ

Remark

The trigonometric relations between the trigonometric functions of the related angles which we studied before are identities because all real values of the variable satisfy them.

For example :

$\sin(\pi - \theta) = \sin\theta$, $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$, ...

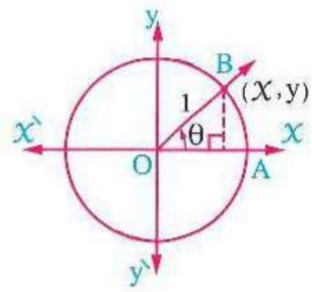
The equation

- It is a true equality for some real values of the variable which satisfy this equality and it is not true for some others which do not satisfy it.

For example :

The equality : $\cos\theta = \sin\theta$ is called equation because it is true for some real values of the variable θ , not for all real values of the variable θ and this because :

From the opposite figure :



From our previous study , we found that :

$\cos\theta = X$, $\sin\theta = y$

$\therefore \cos\theta = \sin\theta$, when $X = y$ only and

this happens when $\theta = 45°$ or $225°$

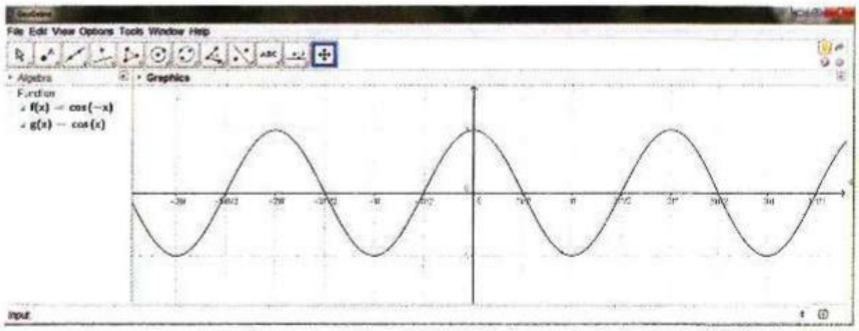
or any of the equivalent angles for them.

Remark

We can determine if the relation represents an identity or an equation and this by the graphical representation to the limits of the two determined functions, if the two functions are intersecting at all points (coincide), then the relation represents an identity and if the two functions are intersecting at some points only, then the relation represents an equation.

For example :

- **In the opposite figure :**



The two functions

$f_1 : f_1(\theta) = \cos(-\theta)$,

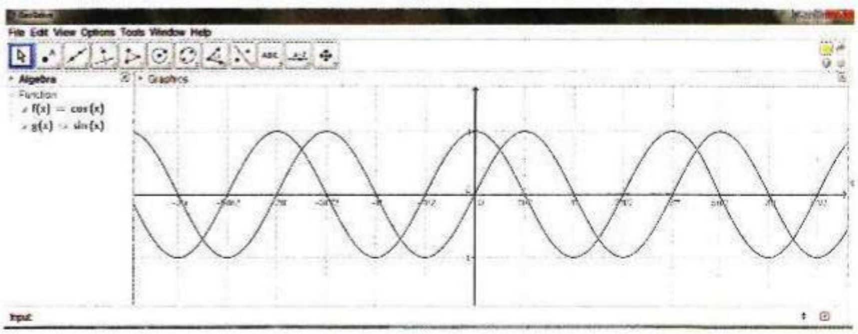
$f_2 : f_2(\theta) = \cos\theta$

are intersecting at all points (coincide)

, then the equality :

$\cos(-\theta) = \cos\theta$ is called identity.

- **In the opposite figure :**



The two functions

$f_1 : f_1(\theta) = \cos X$,

$f_2 : f_2(\theta) = \sin\theta$

are intersecting at some points.

, then the equality :

$\cos\theta = \sin\theta$ is called equation.

Basic trigonometric identities

We studied before the following trigonometric identities :

1 The identity of the trigonometric functions and their reciprocal :

- $\cos \theta = \frac{1}{\sec \theta}$, $\sec \theta = \frac{1}{\cos \theta}$
- $\sin \theta = \frac{1}{\csc \theta}$, $\csc \theta = \frac{1}{\sin \theta}$
- $\tan \theta = \frac{1}{\cot \theta}$, $\cot \theta = \frac{1}{\tan \theta}$

2 The expressing of tan θ and cot θ in terms of sin θ and cos θ :

- $\tan \theta = \frac{\sin \theta}{\cos \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$

3 The identity of the trigonometric functions of two complementary angles :

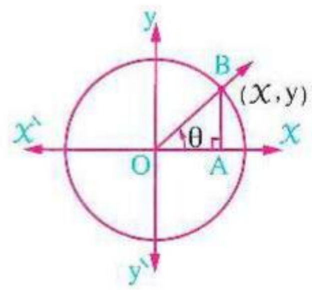
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$, • $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$, • $\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$, • $\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$

4 The identity of the trigonometric functions of the two angles (θ and (– θ)) :

- $\sin(-\theta) = -\sin \theta$, • $\cos(-\theta) = \cos \theta$
- $\csc(-\theta) = -\csc \theta$, • $\sec(-\theta) = \sec \theta$
- $\tan(-\theta) = -\tan \theta$, • $\cot(-\theta) = -\cot \theta$

5 Pythagorian identity :

For any directed angle of measure θ in the standard position , if its terminal side cuts the unit circle at the point (x , y) , then :



$x^2 + y^2 = 1$

$\because \cos \theta = x , \sin \theta = y$

$\therefore \quad \cos^2 \theta + \sin^2 \theta = 1 \qquad (1)$

- Dividing both sides of the relation (1) by $\cos^2 \theta$, we find that :

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \qquad \therefore \quad 1 + \tan^2 \theta = \sec^2 \theta$$

- Dividing both sides of the relation (1) by $\sin^2 \theta$, we find that :

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \qquad \therefore \quad \cot^2 \theta + 1 = \csc^2 \theta$$

Remarks

1. From : $\sin^2\theta + \cos^2\theta = 1$, we get : $\boxed{\sin^2\theta = 1 - \cos^2\theta}$ and $\boxed{\cos^2\theta = 1 - \sin^2\theta}$
2. From : $1 + \tan^2\theta = \sec^2\theta$, we get : $\boxed{\tan^2\theta = \sec^2\theta - 1}$ and $\boxed{\sec^2\theta - \tan^2\theta = 1}$
3. From : $\cot^2\theta + 1 = \csc^2\theta$, we get : $\boxed{\cot^2\theta = \csc^2\theta - 1}$ and $\boxed{\csc^2\theta - \cot^2\theta = 1}$

Check your understanding

Choose the correct answer : $\sin^2\theta + \cos^2\theta \neq$ ....

(a) $\tan\theta \cot\theta$ (b) $\sin^2 2\theta + \cos^2 2\theta$ (c) $\cot^2\theta - \csc^2\theta$ (d) $\sec^2\theta - \tan^2\theta$

Simplifying the trigonometric expressions

We mean by simplifying the trigonometric expression is to put it in the simplest form , by using the basic trigonometric identities.

Example 1

Write each of the following expressions in the simplest form :

1 $\frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$

2 $\sin\left(\frac{\pi}{2} - \theta\right)\csc\theta$

3 $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$

4 $\dfrac{1 + \cot^2\left(3\frac{\pi}{2} - \theta\right)}{1 + \tan^2\left(\frac{3\pi}{2} + \theta\right)}$

Solution

1 $\frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - \tan^2\theta = 1$

2 $\sin\left(\frac{\pi}{2} - \theta\right)\csc\theta = \cos\theta\csc\theta = \frac{\cos\theta}{\sin\theta} = \cot\theta$

Notice that

- $\frac{1}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2 = \sec^2\theta$
- $\frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta$

3 $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = \sin^2\theta + \cancel{2\sin\theta\cos\theta} + \cos^2\theta - \cancel{2\sin\theta\cos\theta}$

$= \sin^2\theta + \cos^2\theta = 1$

Remember that

$(a + b)^2 = a^2 + 2ab + b^2$

4 $\dfrac{1+\cot^2\left(\frac{3\pi}{2}-\theta\right)}{1+\tan^2\left(\frac{3\pi}{2}+\theta\right)} = \dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \dfrac{\sec^2\theta}{\csc^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

Notice that

$\dfrac{\sec^2\theta}{\csc^2\theta} = \dfrac{1}{\cos^2\theta} \div \dfrac{1}{\sin^2\theta}$

$= \dfrac{1}{\cos^2\theta} \times \sin^2\theta$

$= \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

TRY TO SOLVE

Put in the simplest form each of the following expressions :

1 $\dfrac{1}{\sin^2\theta} - \dfrac{1}{\tan^2\theta}$ 2 $\sin\left(\dfrac{\pi}{2}-\theta\right)\sec(2\pi-\theta)$ 3 $\dfrac{1-\sin^2\theta}{\cos^2\theta-1}$

Trigonometric identities

To prove the validity of the trigonometric identity , we follow one of the two methods :

1 Put one of the two sides of the identity in the form of the other side using the basic trigonometric identities.

2 Put the two sides of the trigonometric identity in the simplest form , to prove that the two sides have the same result when they are in the simplest form.

Example 2

Prove the validity of the identity : $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1$

Solution

L.H.S. $= \sin^2\theta - \cos^2\theta = \sin^2\theta - (1-\sin^2\theta)$

$= \sin^2\theta - 1 + \sin^2\theta$

$= 2\sin^2\theta - 1 =$ R.H.S.

Notice that

$\cos^2\theta = 1 - \sin^2\theta$

Example 3

Prove the validity of the identity : $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$

Solution

L.H.S. $= \sin^4\theta - \cos^4\theta = (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$

$= 1 \times (\sin^2\theta - \cos^2\theta)$

$= 1 - \cos^2\theta - \cos^2\theta$

$= 1 - 2\cos^2\theta$

Notice that:

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin^2\theta = 1 - \cos^2\theta$

Example 4

Prove the validity of the identity : $\frac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$

Solution

$$\text{L.H.S.} = \frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} = \frac{(1+\cos\theta)\,\cancel{(1-\cos\theta)}}{\cancel{1-\cos\theta}} = 1 + \cos\theta = \text{R.H.S.}$$

TRY TO SOLVE

Prove the validity of the following identities :

1. $\frac{\cos^2\theta}{1+\sin\theta} = 1 - \sin\theta$
2. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

Example 5

Prove the validity of the identity : $\tan\theta + \cot\theta = \csc\theta\sec\theta$

Solution

$$\begin{aligned}\text{L.H.S.} = \tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\\ &= \frac{1}{\sin\theta\cos\theta}\\ &= \csc\theta\sec\theta\\ &= \text{R.H.S.}\end{aligned}$$

Notice that

To make the proof easy , we write the expression in terms of $\sin\theta$ and $\cos\theta$ only , using the following relations :

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$

Example 6

Prove the validity of the identity : $2\cos^2\theta - 1 = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution

$$\text{R.H.S.} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\sec^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} = \left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)\times\cos^2\theta$$

$$= \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = \cos^2\theta - 1 + \cos^2\theta$$

$$= 2\cos^2\theta - 1 = \text{L.H.S.}$$

Example 7

Prove the validity of the identity : $\sec^2\theta - \tan^2\theta \sin^2\theta = \cos^2\theta + 2\sin^2\theta$

Solution

$$\text{L.H.S.} = \sec^2\theta - \tan^2\theta\sin^2\theta = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \times \sin^2\theta = \frac{1}{\cos^2\theta} - \frac{\sin^4\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^4\theta}{\cos^2\theta} = \frac{\cancel{(1-\sin^2\theta)}\,(1+\sin^2\theta)}{\cancel{1-\sin^2\theta}} = 1 + \sin^2\theta \qquad (1)$$

$$\text{R.H.S.} = \cos^2\theta + 2\sin^2\theta = 1 - \sin^2\theta + 2\sin^2\theta = 1 + \sin^2\theta \qquad (2)$$

From (1) and (2) , we get that : L.H.S. = R.H.S.

TRY TO SOLVE

Prove the validity of the following identity : $\frac{1-\cot^2\theta}{1+\cot^2\theta} = 2\sin^2\theta - 1$

Example 8

If $\sin\theta + \sin(270° - \theta) = \frac{1}{2}$, **find the value of :** $\sin\theta\cos\theta$, where $\theta \in]0 , \frac{\pi}{2}[$

Solution

$\because \sin\theta + \sin(270° - \theta) = \frac{1}{2}$

$\therefore \sin\theta - \cos\theta = \frac{1}{2}$ (squaring both sides)

$\therefore \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = \frac{1}{4}$

$\therefore 1 - 2\sin\theta\cos\theta = \frac{1}{4}$

$\therefore -2\sin\theta\cos\theta = \frac{-3}{4}$

$\therefore \sin\theta\cos\theta = \frac{3}{8}$

Lesson 2

Solving trigonometric equations

- Solving trigonometric equation means finding the values of the variable in the equation which satisfy this equation using the trigonometric identities.

General solution of the trigonometric equation

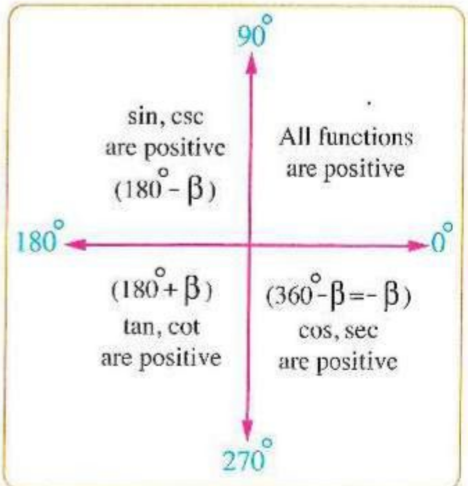
To find the general solution of the trigonometric equation in the form :

$\cos\theta = a$, $\sin\theta = a$ **or** $\tan\theta = a$ **, follow the following steps :**

1 Let β be the measure of the acute angle which satisfies the equation :

$\cos\theta = |a|$, $\sin\theta = |a|$, $\tan\theta = |a|$

2 Determine the quadrant in which the angle lies according to the sign of a "Look to the opposite figure"



3 Find the values of the angle θ where :

- If θ lies in the first quadrant , then $\theta = \beta$
- If θ lies in the second quadrant , then $\theta = 180° - \beta$
- If θ lies in the third quadrant , then $\theta = 180° + \beta$
- If θ lies in the fourth quadrant , then $\theta = 360° - \beta$

4 We add a number of periods $(2\,n\,\pi)$ where $n \in \mathbb{Z}$ to the values of θ to get the general solution of the trigonometric equation.

Remark

- $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all real values of θ

So , we find that the two equations : $\sin\theta = a$, $\cos\theta = a$ don't have solution in $\mathbb{R}$, if $a \notin [-1\ ,1]$

For example :

Each of the equations : $\sin\theta = 1.3$, $\cos\theta = 2.5$, $\sin\theta = -1.4$, $\sec\theta = 0.5$ and $\csc\theta = -0.7$ doesn't have real solutions.

i.e. It is not necessary that there real solutions for every trigonometric equations.

Example 1

Find the general solution of each of the following equations :

1 $\cos\theta = \frac{1}{2}$ **2** $2\sin\theta - \sqrt{2} = 0$ **3** $\sqrt{3}\tan\theta - 1 = 0$

Solution

1 $\cos\theta = \frac{1}{2}$ (positive)

$\therefore$ θ lies in the first quadrant $\therefore \theta = 60°$

or θ lies in the fourth quadrant.

$\therefore \theta = 360° - 60° = 300°$ and it is equivalent to $(-60°)$

Adding $(2\,n\,\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \frac{\pi}{3} + 2\,n\,\pi$ or $\theta = -\frac{\pi}{3} + 2\,n\,\pi$

$\therefore$ The general solution of the equation is : $\pm\frac{\pi}{3} + 2\,n\,\pi$, where $n \in \mathbb{Z}$

2 $\sin\theta = \frac{\sqrt{2}}{2}$ (positive)

$\therefore$ θ lies in the first quadrant $\therefore \theta = 45°$

or θ lies in the second quadrant

$\therefore \theta = 180° - 45° = 135°$

Adding $(2\,n\,\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \frac{\pi}{4} + 2\,n\,\pi$ or $\theta = \frac{3}{4}\pi + 2\,n\,\pi$

$\therefore$ The general solution of the equation is : $\theta = \frac{\pi}{4} + 2\,n\,\pi$ or $\theta = \frac{3}{4}\pi + 2\,n\,\pi$, where $n \in \mathbb{Z}$

3 $\tan\theta = \frac{1}{\sqrt{3}}$ (positive)

$\therefore$ θ lies in the first quadrant $\qquad \therefore \theta = 30°$

or θ lies in the third quadrant $\qquad \therefore \theta = 180° + 30° = 210°$

Adding $(2\,n\,\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \frac{\pi}{6} + 2\,n\,\pi$ or $\theta = \frac{7}{6}\pi + 2\,n\,\pi$

$\therefore$ The general solution of the equation is : $\theta = \frac{\pi}{6} + 2\,n\,\pi$ or $\theta = \frac{7}{6}\pi + 2\,n\,\pi$, where $n \in \mathbb{Z}$

and we can write the general solution of the equation in another more simple form as the following :

The general solution of the equation is : $\theta = \frac{\pi}{6} + n\,\pi$ where $n \in \mathbb{Z}$

and this by adding $n\,\pi$ to the smallest positive measure.

Remark

From the previous , we can deduce that :

If β is the smallest positive measure satisfies the equation , $n \in \mathbb{Z}$, then :

1. The general solution of the equation $\sin\theta = a$ is $\theta = \beta + 2\,\pi\,n$, $\theta = (\pi - \beta) + 2\,\pi\,n$ and that could be written as : $\theta = (-1)^n \times \beta + \pi\,n$
2. The general solution of the equation $\cos\theta = a$ is $\theta = \pm\,\beta + 2\,\pi\,n$
3. The general solution of the equation $\tan\theta = a$ is $\theta = \beta + \pi\,n$

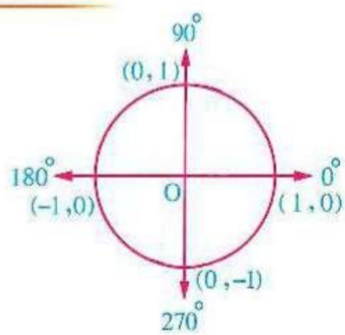
Example 2

Find the general solution of each of the following equations :

1 $\sin\theta = 0$ $\qquad$ **2** $\cos\theta = 0$

3 $\sin\theta = 1$ $\qquad$ **4** $\cos\theta = -1$

Solution



1 $\sin\theta = 0$

$\therefore \theta = 0°$ or $\theta = 180°$

Adding $(2\,\pi\,n)$ where $n \in \mathbb{Z}$ to the values of θ

$\therefore$ The general solution of the equation is :

$\theta = 2\,\pi\,n$ or $\theta = \pi + 2\,\pi\,n$ where $n \in \mathbb{Z}$

and we can write the general solution of the equation in another more simple form as the following :

The general solution of the equation is : $\theta = \pi\,n$ where $n \in \mathbb{Z}$

2 $\cos\theta = 0$ $\therefore \theta = 90°$ or $\theta = 270°$

Adding $(2\pi n)$ where $n \in \mathbb{Z}$ to the values of θ

$\therefore$ The general solution is : $\theta = \frac{\pi}{2} + 2\pi n$

or $\theta = \frac{3}{2}\pi + 2\pi n$ where $n \in \mathbb{Z}$

and we can write the general solution of the equation in another more simple form as the following :

The general solution is : $\theta = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$

3 $\sin\theta = 1$ $\therefore \theta = 90°$

$\therefore$ The general solution is : $\theta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

4 $\cos\theta = -1$ $\therefore \theta = 180°$

$\therefore$ The general solution is : $\theta = \pi + 2\pi n$ where $n \in \mathbb{Z}$

Remark

From the previous , we can deduce the general solution of the trigonometric equations of the quadrantal angles :

The equation	The general solution
• $\sin\theta = 0$	$\theta = \pi n$
• $\sin\theta = 1$	$\theta = \frac{\pi}{2} + 2\pi n$
• $\sin\theta = -1$	$\theta = \frac{3\pi}{2} + 2\pi n$

The equation	The general solution
• $\cos\theta = 0$	$\theta = \frac{\pi}{2} + \pi n$
• $\cos\theta = 1$	$\theta = 2\pi n$
• $\cos\theta = -1$	$\theta = \pi + 2\pi n$

Example 3

Find the general solution of each of the following equations :

1 $\cot\theta + 1 = 0$ 2 $\cos^2\theta - \cos\theta = 0$

Solution

Notice that

The measure of the acute angle which satisfies that : $\tan\theta = |-1|$ is 45°

1 $\cot\theta = -1$ $\therefore \tan\theta = -1$ (negative)

$\therefore \theta$ lies in the second quadrant. $\therefore \theta = 180° - 45° = 135°$

or θ lies in the fourth quadrant. $\therefore \theta = 360° - 45° = 315°$

Adding $(n\pi)$ where $n \in \mathbb{Z}$ to the smallest positive measure which satisfies the equation "135°"

$\therefore$ The general solution is : $\frac{3}{4}\pi + n\pi$

2 $\cos\theta(\cos\theta - 1) = 0$

$\therefore \cos\theta = 0$

$\therefore \theta = 90°$ or $\theta = 270°$ and it is equivalent to $(-90°)$

Adding $(2n\pi)$ where $n \in \mathbb{Z}$ to the values of θ :

$\therefore \theta = \pm 90° + 2n\pi$

or $\cos\theta = 1$ $\quad\therefore \theta = 0$

Adding $(2n\pi)$ where $n \in \mathbb{Z}$:

$\therefore \theta = 2n\pi$

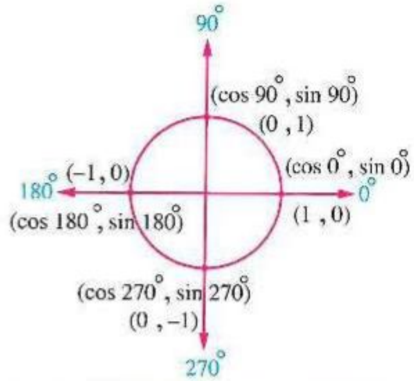
$\therefore$ The general solution is : $\theta = \pm\frac{\pi}{2} + 2n\pi$

or $\theta = 2n\pi$

Remark

Use the unit circle to determine the values of θ when :

$\cos\theta = 0$, $\cos\theta = 1$



TRY TO SOLVE

Find the general solution of each of the following equations :

1 $2\sin\theta - 1 = 0$ **2** $2\cos\theta + \sqrt{3} = 0$ **3** $\tan\theta - \sqrt{3} = 0$

Example 4

Find the general solution of the equation : $\sin\theta\cos\theta = \frac{1}{2}\sin\theta$

Solution

$\because \sin\theta\cos\theta - \frac{1}{2}\sin\theta = 0$ $\quad\therefore \sin\theta\left(\cos\theta - \frac{1}{2}\right) = 0$

$\therefore \sin\theta = 0$ $\quad\therefore \theta = 0°$ $\quad$ or $\theta = 180°$

$\therefore \theta = n\pi$ where $n \in \mathbb{Z}$

or $\cos\theta - \frac{1}{2} = 0$ $\quad\therefore \cos\theta = \frac{1}{2}$(positive)

$\therefore \theta$ lies in the first quadrant. $\quad\therefore \theta = 60°$

or θ lies in the fourth quadrant. $\quad\therefore \theta = 360° - 60° = 300°$ and it is equivalent to $(-60°)$

$\therefore \theta = \pm\frac{\pi}{3} + 2n\pi$ where $n \in \mathbb{Z}$

$\therefore$ The general solution is : $\theta = n\pi$ or $\theta = \pm\frac{\pi}{3} + 2n\pi$ where $n \in \mathbb{Z}$

TRY TO SOLVE

Find the general solution of the equation : $2\sin\theta\cos\theta - \sqrt{3}\sin\theta = 0$

Solving the trigonometric equation in the interval $[0 , 2\pi[$

Example 5

If $\theta \in [0 , 360^\circ[$ Find the solution set of each of the following equations :

1 $2\cos\theta + 1 = 0$ 2 $\sqrt{2}\sec\theta - 2 = 0$

Solution

1 $\because 2\cos\theta + 1 = 0 \quad \therefore \cos\theta = -\frac{1}{2}$ (negative)

$\therefore$ θ lies in the second quadrant or in the third quadrant.

, $\because$ the acute angle of cosine $= \frac{1}{2}$, its measure is 60°

$\therefore \theta = 180^\circ - 60^\circ = 120^\circ$ or $\theta = 180^\circ + 60^\circ = 240^\circ$

$\therefore$ The S.S. $= \{120^\circ , 240^\circ\}$

2 $\because \sqrt{2}\sec\theta - 2 = 0 \qquad \therefore \sec\theta = \frac{2}{\sqrt{2}}$

$\therefore \cos\theta = \frac{\sqrt{2}}{2}$ (positive)

$\therefore$ θ lies in the first quadrant or in the fourth quadrant.

, $\because$ the acute angle of cosine $= \frac{\sqrt{2}}{2}$, its measure is 45°

$\therefore \theta = 45^\circ$ or $\theta = 360^\circ - 45^\circ = 315^\circ$

$\therefore$ The S.S. $= \{45^\circ , 315^\circ\}$

Example 6

Find the solution set of the equation : $4\cos^2\theta - 3 = 0$, where $\theta \in [0 , 360^\circ[$

Solution

$\because 4\cos^2\theta - 3 = 0 \qquad \therefore 4\cos^2\theta = 3 \qquad \therefore \cos^2\theta = \frac{3}{4}$

$\therefore \cos\theta = \pm\frac{\sqrt{3}}{2} \qquad \therefore \cos\theta = \frac{\sqrt{3}}{2}$ (positive)

$\therefore$ θ lies in the first quadrant or in the fourth quadrant.

, $\because$ the acute angle of cosine $= \frac{\sqrt{3}}{2}$, its measure is 30°

$\therefore \theta = 30^\circ$ or $\theta = 360^\circ - 30^\circ = 330^\circ$

or $\cos\theta = \frac{-\sqrt{3}}{2}$ (negative)

$\therefore$ θ lies in the second quadrant or in the third quadrant.

$\therefore \theta = 180^\circ - 30^\circ = 150^\circ$ or $\theta = 180^\circ + 30^\circ = 210^\circ$

$\therefore$ The S.S. $= \{30^\circ , 150^\circ , 210^\circ , 330^\circ\}$

TRY TO SOLVE

Find the solution set of each of the following equations where $\theta \in [0 , 2\pi[$

1 $\sqrt{2} \csc \theta - 2 = 0$ **2** $\tan^2 \theta = 1$

Example 7

Find the solution set of the equation : $2 \sin \theta \cos \theta + 3 \cos \theta = 0$, where $\theta \in [0 , \pi[$

Solution

$\because 2 \sin \theta \cos \theta + 3 \cos \theta = 0$

$\therefore \cos \theta (2 \sin \theta + 3) = 0$

$\therefore \cos \theta = 0$

$\therefore \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (refused because $\theta \in [0 , \pi[$)

or $2 \sin \theta + 3 = 0$

$\therefore \sin \theta = \frac{-3}{2}$ (this equation has no solution because $-1 \le \sin \theta \le 1$)

$\therefore$ The S.S. $= \left\{\frac{\pi}{2}\right\}$

Example 8

Find the solution set of the equation : $4 \sin^2 \theta - 3 \sin \theta \cos \theta = 0$, where $\theta \in [0 , 360^\circ[$

Solution

$\because 4 \sin^2 \theta - 3 \sin \theta \cos \theta = 0$

$\therefore \sin \theta (4 \sin \theta - 3 \cos \theta) = 0$

$\therefore \sin \theta = 0$

$\therefore \theta = 0^\circ$ or $\theta = 180^\circ$

or $4 \sin \theta - 3 \cos \theta = 0$

$\therefore 4 \sin \theta = 3 \cos \theta$

$\therefore \frac{\sin \theta}{\cos \theta} = \frac{3}{4}$

$\therefore \tan \theta = \frac{3}{4}$ (positive)

$\therefore \theta$ lies in the first quadrant or in the third quadrant.

, $\because$ the acute angle of tangent $= \frac{3}{4}$, its measure is $36^\circ\, 52'$

$\therefore \theta = 36^\circ\, 52'$ or $\theta = 180^\circ + 36^\circ\, 52' = 216^\circ\, 52'$

$\therefore$ The S.S. $= \{0^\circ , 36^\circ\, 52' , 180^\circ , 216^\circ\, 52'\}$

TRY TO SOLVE

If $0° < \theta < 360°$ Find the solution set of the equation : $2 \sin \theta \cos \theta = 3 \cos^2 \theta$

Example 9

Find the solution set of the equation : $2 \sin^2 \theta - \cos \theta - 1 = 0$, where $\theta \in [0 , 360° [$

Solution

$\because \sin^2 \theta = 1 - \cos^2 \theta$

$\therefore 2 (1 - \cos^2 \theta) - \cos \theta - 1 = 0$

$\therefore 2 - 2 \cos^2 \theta - \cos \theta - 1 = 0$

$\therefore 1 - \cos \theta - 2 \cos^2 \theta = 0$

$\therefore (1 + \cos \theta) (1 - 2 \cos \theta) = 0$

$\therefore 1 + \cos \theta = 0$

$\therefore \cos \theta = -1$

$\therefore \theta = 180°$

or $1 - 2 \cos \theta = 0$

$\therefore \cos \theta = \frac{1}{2}$ (positive)

$\therefore \theta$ lies in the first quadrant or in the fourth quadrant.

, $\because$ the acute angle of cosine $= \frac{1}{2}$, its measure is $60°$

$\therefore \theta = 60°$ or $\theta = 360° - 60° = 300°$

$\therefore$ The S.S. $= \{60° , 180° , 300°\}$

Using the technology

From example (1) , we found that :

The general solution of the equation : $\cos \theta = \frac{1}{2}$ is $\theta = \pm \frac{\pi}{3} + 2 n \pi$, where $n \in \mathbb{Z}$

We can verify the solution by drawing the two function :

$f_1 : f_1 (\theta) = \cos \theta$, $f_2 : f_2 (\theta) = \frac{1}{2}$

by using one of the drawing programs and determining the corresponding values of θ for the intersection points of the two functions and compare them with the values of θ in the general solution at $n = ... , -2 , -1 , 0 , 1 , 2 , ...$